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TREATISE

ON

TRILINEAR CO-ORDINATES,

INTENDED CHIEFLY FOR THE USE OF

JUNIOR STUDENTS.

BY

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PRELIMINARY CHAPTER.

A.

To shew that if $\frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z}$; *then*

$$\frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z} = \frac{l\alpha + m\beta + n\gamma}{lx + my + nz}.$$

Let each of the given fractions $= \lambda$; we shall have

$$\begin{aligned} \alpha &= \lambda x, \\ \beta &= \lambda y, \\ \gamma &= \lambda z; \end{aligned} \qquad \text{(i)}$$

and, multiplying these equations by l, m, n, respectively, and adding, we get

$$\begin{aligned} l\alpha + m\beta + n\gamma &= \lambda lx + \lambda my + \lambda nz \\ &= \lambda(lx + my + nz); \end{aligned}$$

whence, $$\frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z} = \lambda = \frac{l\alpha + m\beta + n\gamma}{lx + my + nz}.$$

Examples of the theorem here proved occur in Arts. 20, 23, 25, 31, 34, 54, etc.

B.

To shew that if $\frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z}$; *then*

$$\frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z} = \left(\frac{l\beta\gamma + m\gamma\alpha + n\alpha\beta}{lyz + mzx + nxy}\right)^{\frac{1}{2}}.$$

Multiplying equations (i) two and two together, we get

$$\begin{aligned} \beta\gamma &= \lambda^2 yz, \\ \gamma\alpha &= \lambda^2 zx, \\ \alpha\beta &= \lambda^2 xy; \end{aligned} \qquad \text{(ii)}$$

and multiplying these by l, m, n, respectively, and adding, we have

$$\begin{aligned} l\beta\gamma + m\gamma\alpha + n\alpha\beta &= \lambda^2 lyz + \lambda^2 mzx + \lambda^2 nxy \\ &= \lambda^2 (lyz + mzx + nxy). \end{aligned}$$

Therefore

$$\frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z} = \lambda = \left(\frac{l\beta\gamma + m\gamma\alpha + n\alpha\beta}{lyz + mzx + nxy}\right)^{\frac{1}{2}}.$$

The reader will find instances of the application of this theorem in Arts. 40, 45, etc.

C.

To shew that if $\frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z}$; *then*

$$\frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z} = \lambda = \left(\frac{l\alpha^2 + m\beta^2 + n\gamma^2}{lx^2 + my^2 + nz^2}\right)^{\frac{1}{2}}.$$

Squaring equations (i), we have

$$\begin{aligned} \alpha^2 &= \lambda^2 x^2, \\ \beta^2 &= \lambda^2 y^2, \\ \gamma^2 &= \lambda^2 z^2; \end{aligned} \qquad \text{(iii)}$$

and multiplying these respectively by l, m, n, and adding, we get

$$\begin{aligned} l\alpha^2 + m\beta^2 + n\gamma^2 &= \lambda^2 lx^2 + \lambda^2 my^2 + \lambda^2 nz^2 \\ &= \lambda^2 (lx^2 + my^2 + nz^2). \end{aligned}$$

Hence,

$$\frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z} = \lambda = \left(\frac{l\alpha^2 + m\beta^2 + n\gamma^2}{lx^2 + m\beta^2 + n\gamma^2}\right)^{\frac{1}{2}}.$$

This theorem is employed in Art. 45.

D.

The discriminant of the quadric

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2D\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta$$

[denoted hereafter by $\phi(\alpha, \beta, \gamma)$,] or the condition that it should be resolvable into linear factors, is obtained (Art. 147) by the elimination of α, β, γ between the equations

$$\left(\frac{d\phi}{d\alpha}\right) = 0, \quad \left(\frac{d\phi}{d\beta}\right) = 0, \quad \left(\frac{d\phi}{d\gamma}\right) = 0;$$

i.e.: between

$$A\alpha + F\beta + E\gamma = 0,$$

$$F\alpha + B\beta + D\gamma = 0,$$

$$E\alpha + D\beta + C\gamma = 0.$$

It is, therefore,

$$\begin{vmatrix} A, & F, & E \\ F, & B, & D \\ E, & D, & C \end{vmatrix} = 0. \qquad \text{(iv)}$$

The determinant which forms the left-hand member of equation (iv) occurs frequently in the course of the present work. It is convenient, therefore, to have some abbreviation for it. We shall denote it by Δ, and wherever this symbol is employed (see Arts. 138, 144, 146, 162, etc.) the above meaning is to be attached to it.

The abbreviation (A, B, C) in which only the diagonal constituents are given is sometimes (Art. 167) used to represent the same determinant.

The *first minors* of this determinant [viz. the determinants, with proper signs, which we obtain by cancelling successively the column and row to which the several constituents A, B, C, D, E, F are common], are also of frequent occurrence: these, therefore, it will be convenient to denote by A', B', C', D', E', F', respectively; so that

$$\begin{vmatrix} B, & D \\ D, & C \end{vmatrix} = BC - D^2 = A', \quad -\begin{vmatrix} A, & E \\ F, & D \end{vmatrix} = EF - AD = D',$$

$$\begin{vmatrix} A, & E \\ E, & C \end{vmatrix} = CA - E^2 = B', \quad \begin{vmatrix} F, & B \\ E, & D \end{vmatrix} = FD - BE = E',$$

$$\begin{vmatrix} A, & E \\ F, & B \end{vmatrix} = AB - F^2 = C', \quad -\begin{vmatrix} F, & D \\ E, & C \end{vmatrix} = DE - CF = F'.$$

The value of Δ may be expressed in terms of these minors as follows:—

$$\Delta = AA' + FF' + EE',$$

$$\Delta = FF' + BB' + DD',$$

$$\Delta = EE' + DD' + CC'.$$

An instance of the use of this notation will be seen in Art. 138.

If a determinant be of the form

$$\begin{vmatrix} a_1 + l_1, & b_1 + m_1, & c_1 + n_1 \\ a_2 + l_2, & b_2 + m_2, & c_2 + n_2 \\ a_3 + l_3, & b_3 + m_3, & c_3 + n_3 \end{vmatrix}$$

or $(a_1 + l_1,\ b_2 + m_2,\ c_3 + n_3)$, each constituent being the sum of two others, the determinant is equal to the sum of all the determinants which can be formed by taking for each column one of the partial columns of the corresponding column of the original determinant; that is to say,

$$\begin{aligned}(a_1 + l_1, b_2 + m_2, c_3 + n_3) = (a_1\ b_2\ c_3) &+ (a_1 m_2 c_3) + (a_1 b_2 n_3) \\ &+ (a_1 m_2 n_3) + (l_1\ b_2 c_3) + (b_1 m_2 c_3) \\ &+ (l_1\ b_2 n_3) + (l_1 m_2 n_3).\end{aligned}$$

An application of this property will be found in Art. 167.

E.

The bordered determinant

$$\begin{vmatrix} A, & F, & E, & l \\ F, & B, & D, & m \\ E, & D, & C, & n \\ l, & m, & n, & 0 \end{vmatrix} \quad \text{or,} \quad \begin{vmatrix} 0, & l, & m, & n \\ l, & A, & F, & E \\ m, & F, & B, & D \\ n, & E, & D, & C \end{vmatrix}$$

will be denoted by ${}^{l}\underset{n}{\Delta}{}^{m}$.

It becomes, on expansion,

$$-l \begin{vmatrix} l, & m, & n \\ F, & B, & D \\ E, & D, & C \end{vmatrix} + m \begin{vmatrix} l, & m, & n \\ A, & F, & E \\ E, & D, & C \end{vmatrix} - n \begin{vmatrix} l, & m, & n \\ A, & F, & E \\ F, & B, & D \end{vmatrix},$$

that is,

$$-l \begin{vmatrix} l, & F, & E \\ m, & B, & D \\ n, & D, & C \end{vmatrix} + m \begin{vmatrix} l, & A, & E \\ m, & F, & D \\ n, & E, & C \end{vmatrix} - n \begin{vmatrix} l, & A, & F \\ m, & F, & B \\ n, & E, & D \end{vmatrix},$$

or,

$$-\begin{vmatrix} B, & D \\ D, & C \end{vmatrix} l^2 - \begin{vmatrix} E, & C \\ A, & E \end{vmatrix} m^2 - \begin{vmatrix} A. & F \\ F, & B \end{vmatrix} n^2 - 2 \begin{vmatrix} A, & F \\ E, & D \end{vmatrix} mn - \text{etc.} \ldots$$

or, if we employ the notation explained in (*D*),

$$-(A'l^2 + B'm^2 + C'n^2 + 2D'mn + 2E'nl + 2F'lm).$$

The expression within the brackets, from its similarity in form to the quadric $\phi(\alpha, \beta, \gamma)$, will be denoted by $\phi(l, m, n)'$; we shall have, therefore,

$$\begin{vmatrix} A, & F, & E, & l \\ F, & B, & D, & m \\ E, & D, & C, & n \\ l, & m, & n, & 0 \end{vmatrix} = {}^{l}\Delta^{m}_{n} = -\phi(l, m, n)'.$$

F.

If $\phi(\alpha, \beta, \gamma)$ *represent the general quadric*, as in section (D) of this chapter; *then*

$$\left(\frac{d\phi}{d\alpha}\right)\alpha_1 + \left(\frac{d\phi}{d\beta}\right)\beta_1 + \left(\frac{d\phi}{d\gamma}\right)\gamma_1 = \left(\frac{d\phi}{d\alpha_1}\right)\alpha + \left(\frac{d\phi}{d\beta_1}\right)\beta + \left(\frac{d\phi}{d\gamma_1}\right)\gamma.$$

For,

$$\begin{aligned}\left(\frac{d\phi}{d\alpha}\right)\alpha_1 + \left(\frac{d\phi}{d\beta}\right)\beta_1 + \left(\frac{d\phi}{d\gamma}\right)\gamma_1 = \quad & (A\alpha + F\beta + E\gamma)\alpha_1 \\ & +(F\alpha + B\beta + D\gamma)\beta_1 \\ & +(E\alpha + D\beta + C\gamma)\gamma_1, \\ = \quad & (A\alpha_1 + F\beta_1 + E\gamma_1)\alpha \\ & +(F\alpha_1 + B\beta_1 + D\gamma_1)\beta \\ & +(E\alpha + D\beta_1 + C\gamma_1)\gamma. \\ = & \left(\frac{d\phi}{d\alpha_1}\right)\alpha + \left(\frac{d\phi}{d\beta_1}\right)\beta + \left(\frac{d\phi}{d\gamma_1}\right)\gamma.\end{aligned}$$

TRILINEAR CO-ORDINATES.

CHAPTER I.

EXPLANATION OF THE METHOD. THE STRAIGHT LINE.

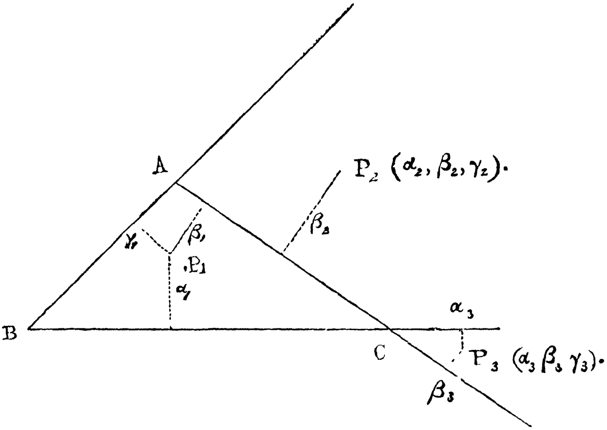
1. THE position of a point in Trilinear Co-ordinates is given by means of its perpendicular distances from three fixed straight lines which do not meet in a point. In accordance with the usual Trigonometrical notation, A, B, C are employed to denote the angles of this *triangle of reference*, and a, b, c the sides which respectively subtend them; the area of the triangle being represented by S.

The perpendicular distances of any point from the three sides are called the *co-ordinates* of the point, and are denoted by the Greek letters α, β, γ; and the point itself, for the sake of brevity, by (α, β, γ). Thus $\left(\frac{2S}{a}, 0, 0\right)$, $\left(0, \frac{2S}{b}, 0\right)$, $\left(0, 0, \frac{2S}{c}\right)$, and $\left(0, \frac{a}{2}\sin C, \frac{a}{2}\sin B\right)$, $\left(\frac{b}{2}\sin C, 0, \frac{b}{2}\sin A\right)$, $\left(\frac{c}{2}\sin B, \frac{c}{2}\sin A, 0\right)$, represent respectively the three vertices of the triangle of reference, and the middle points of the opposite sides.

2. It is necessary, as in the Cartesian system, to lay down some rule for the interpretation of signs; the position of a point depending upon the *sign*, no less than upon the *numerical value* of its co-ordinates. The *direction* of a perpendicular is indicated by the one, as its absolute *length* is by the other. Thus, α and $-\alpha$ refer to points which are equidistant from the side BC, but which lie on opposite sides of that line.

It is customary to regard the α- co-ordinate of any point P as *positive* or *negative*, according as the vertex A and the point P lie on the same or on opposite sides of BC; the sign of the β- and γ- co-ordinates being determined in a similar manner. If the point lie *on* the side BC, it is obvious that its α- co-ordinate will be $= 0$. And similarly for points which lie on either of the other two sides.

Fig. 1.


A
P_2 $(\alpha_2, \beta_2, \gamma_2)$.
γ_1
β_1
P_1
α_1
β_2
α_3
B
C
P_3 $(\alpha_3, \beta_3, \gamma_3)$.
β_3

Thus then the co-ordinates $(\alpha_1, \beta_1, \gamma_1)$ of any point P_1, *within* the triangle of reference, are all positive; whilst in the figure P_2 has *one* (viz. the β- co-ordinate), and P_3 *two* (viz. the β- and γ- co-ordinates) negative.

3. Bearing in mind this convention with regard to signs, the student will have no difficulty in proving for himself that *the co-ordinates of any point* (α, β, γ) *satisfy the relation*

$$a\alpha + b\beta + c\gamma = 2S. \tag{1}$$

For if the point (α, β, γ), whatever its position, be joined to the vertices of the triangle of reference, it will be seen at once that the algebraical sum of the triangular areas represented by $\frac{1}{2}a\alpha$, $\frac{1}{2}b\beta$, $\frac{1}{2}c\gamma$, is always equal to the area of the triangle ABC.

4. By means of the relation (1) any trilinear expression may be

rendered homogeneous, or raised to any required order. For, since the quantity $\frac{a\alpha + b\beta + c\gamma}{2S}$ is equal to unity, we can, without altering the value of the expression, multiply any term by such a power of it as may be necessary to raise it to the given dimensions.

Thus the equation

$$\alpha^2 + h(\beta + \gamma) + k^2 = 0$$

may be written in the homogeneous form

$$4S^2\alpha^2 + 2Sh(a\alpha + b\beta + c\gamma)(\beta + \gamma) + k^2(a\alpha + b\beta + c\gamma)^2 = 0,$$

and the linear expression

$$l\alpha + m\beta + n\gamma$$

replaced by the equivalent quadric

$$\frac{1}{2S}(a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma),$$

or $\frac{1}{2S}\left\{ al\alpha^2 + bm\beta^2 + cn\gamma^2 + (bn+cm)\beta\gamma + (cl+an)\gamma\alpha + (am+bl)\alpha\beta \right\}.$

5. The relation of Art. 3 is also useful if it is required to determine the co-ordinates of a point, when their ratios only are given. Instances of its being so employed will be met with in the present chapter (Arts. 20, 23, 25).

6. The foregoing account of the method will suffice for the interpretation of the following simple forms of trilinear equations.

(i.) $\alpha = 0$ represents the locus of points the perpendicular distances of which from BC are $= 0$. This being true only of points which lie on that line of reference, $\alpha = 0$ must be the equation of BC. Similarly $\beta = 0$, $\gamma = 0$, are the equations of the sides CA and AB respectively.

Hence the three lines of reference are sometimes conveniently denoted by the letters α, β, γ; and their points of intersection by $(\beta\gamma)$, $(\gamma\alpha)$, $(\alpha\beta)$.

(ii.) Again $\alpha = k$ (a constant) is the equation of a straight line parallel to BC, at a distance k from it; for it is true only of points which lie on such a line. In the same way $\beta =$ a constant, $\gamma =$ a constant, represent straight lines parallel to CA, AB respectively.

(iii.) The equation $\beta - \gamma = 0$, or $\beta = \gamma$, is true only for points which are equidistant from CA, AB, the perpendiculars upon those sides being either both positive or both negative; whereas $\beta + \gamma = 0$ is the equation of the locus of points whose distances from the same two sides are equal but of opposite sign. Hence $\beta - \gamma = 0$, $\beta + \gamma = 0$ are respectively the equations of the internal and external bisectors of the angle at A; and the equation $\beta^2 - \gamma^2 = 0$ represents this pair of bisectors.

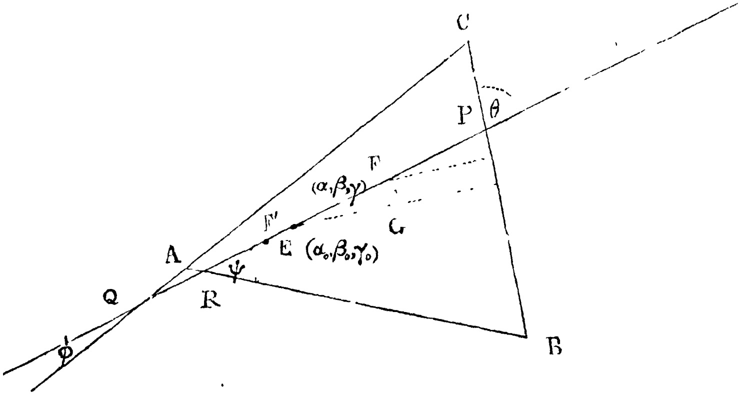
Similarly $\gamma^2 - \alpha^2 = 0$, $\alpha^2 - \beta^2 = 0$ represent the internal and external bisectors of the angles at B and C.

(iv.) $\beta\gamma - \alpha^2 = 0$ is the equation of the locus of points such that the squares of their perpendicular distances from one of the sides of the triangle of reference are equal to the products of their distances from the other two sides. The nature of this locus will be determined hereafter (Art. 216. (R).).

7. *On the general equation of a straight line.*

Let PQR be any straight line, passing through the fixed point

Fig. 2.



$E(\alpha_0, \beta_0, \gamma_0)$, and making with the sides of the triangle of reference the *acute* angles θ, ϕ, ψ.

Take $F(\alpha, \beta, \gamma)$ any other point on the line and let the perpendiculars α, α_0 be drawn. Then, if r be the distance between the two points, we shall have

$$\begin{aligned}\alpha_0 - \alpha &= EG \\ &= r \sin \theta,\end{aligned}$$

and, therefore,

$$\alpha - \alpha_0 = - r \sin \theta.$$

In the same way, we get

$$\begin{aligned}\beta - \beta_0 &= r \sin \phi, \\ \gamma - \gamma_0 &= r \sin \psi.\end{aligned}$$

Hence,

$$\frac{\alpha - \alpha_0}{-\sin \theta} = \frac{\beta - \beta_0}{\sin \phi} = \frac{\gamma - \gamma_0}{\sin \psi} = \pm r \qquad (2)$$

is the equation of the straight line in terms of the co-ordinates of a fixed point on it, and the acute angles which it makes with the sides of the triangle of reference.

The lower sign must be taken with r if (α, β, γ) be on the other side of E, as at F'.

8. It will be observed that the denominators of (2) have not all the same sign: the reader will do well to remember the following rule with respect to them.

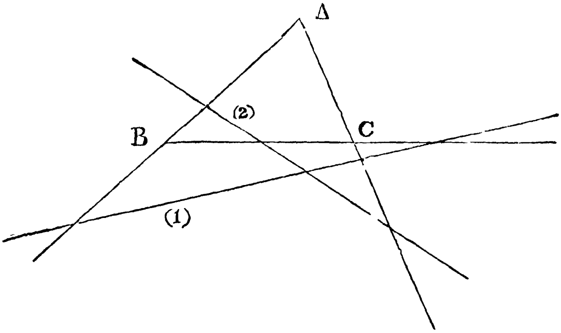
One of the segments QR, RP, PQ of the line always subtends (as does QP in the above figure) the *supplement* of the corresponding vertical angle, instead of the angle itself; and it will be found that in all cases those members of the equation have their denominators of the same sign which correspond to segments that are in this respect similar.

Thus the lines marked (1), (2) in the annexed figure have their equations respectively of the forms

$$\frac{\alpha - \alpha_0}{\sin \theta} = \frac{\beta - \beta_0}{-\sin \phi} = \frac{\gamma - \gamma_0}{\sin \psi},$$

and $$\frac{\alpha - \alpha_0}{\sin \theta} = \frac{\beta - \beta_0}{\sin \phi} = \frac{\gamma - \gamma_0}{-\sin \psi}.$$

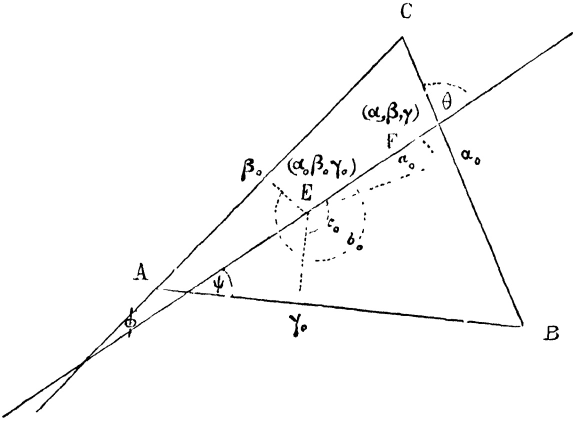
Fig. 3.



We have given this form of the equation of the right line, because, owing to the simplicity of the angles involved, it is more easily written down in particular cases than the form which we now proceed to give, and which, on account of its greater symmetry, we shall hereafter employ.

9. Let E, F, as before, be the points $(\alpha_0, \beta_0, \gamma_0)$, (α, β, γ) respectively, and $EF = r$.

Fig. 4.



Let a_0, b_0, c_0 be the angles which the line EF makes with the perpendiculars from any internal point, as $(\alpha_0, \beta_0, \gamma_0)$; the angles being measured in one and the same direction (namely in that in which we pass from α_0 to β_0, from β_0 to γ_0, and from γ_0 to α_0), as indicated by the dotted lines in the figure.

We shall then have

$$\sin\theta = \cos a_0,\ \sin\phi = \cos(b_1 - \pi) = -\cos b_1,\ \sin\psi = -\cos c_1;$$

and the equation of the last Article becomes, on changing the sign of the whole

$$\frac{\alpha - \alpha_0}{\cos a_0} = \frac{\beta - \beta_0}{\cos b_0} = \frac{\gamma - \gamma_0}{\cos c_0} = \pm r, \qquad (3)$$

which is therefore *the equation of the straight line in terms of the co-ordinates of a fixed point on it and* what we shall in future call *the direction-cosines of the line.*

These direction-cosines we shall occasionally denote by λ, μ, ν, and the straight line itself by (λ, μ, ν), or by $(\cos a_0, \cos b_0, \cos c_0)$.

10. The student would do well, by employing a variety of figures, to satisfy himself that the forms of the equation given in the preceding Articles hold in all cases, whatever be the position chosen for the line itself, or for the points $(\alpha_0, \beta_0, \gamma_0)$, (α, β, γ).

11. *To shew that the equation of the right line may be written in the form*

$$l\alpha + m\beta + n\gamma = 0.$$

Let the direction-cosines of the right line be λ, μ, ν; and its equations

$$\frac{\alpha - \alpha_0}{\lambda} = \frac{\beta - \beta_0}{\mu} = \frac{\gamma - \gamma_0}{\nu}.$$

From these we get

$$\mu\gamma - \nu\beta = \gamma_0\mu - \beta_0\nu,$$
$$\nu\alpha - \lambda\gamma = \alpha_0\nu - \gamma_0\lambda,$$

which give

$$(\gamma_0\lambda - \alpha_0\nu)(\mu\gamma - \nu\beta) = (\gamma_0\mu - \beta_0\nu)(\lambda\gamma - \nu\alpha);$$

whence, multiplying out, dividing by ν, and arranging the terms, we get

$$(\beta_0\nu - \gamma_0\mu)\,\alpha + (\gamma_0\lambda - \alpha_0\nu)\,\beta + (\alpha_0\mu - \beta_0\lambda)\,\gamma = 0,$$

or
$$\begin{vmatrix} \beta_0, & \gamma_0 \\ \mu, & \nu \end{vmatrix} \alpha + \begin{vmatrix} \gamma_0, & \alpha_0 \\ \nu, & \lambda \end{vmatrix} \beta + \begin{vmatrix} \alpha_0, & \beta_0 \\ \lambda, & \mu \end{vmatrix} \gamma = 0, \qquad (4)$$

an equation of the required form. It may be written in the form of a determinant, thus:

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_0, & \beta_0, & \gamma_0 \\ \lambda, & \mu, & \nu \end{vmatrix} = 0. \qquad (5)$$

12. *To shew conversely that every equation of the form*

$$l\alpha + m\beta + n\gamma = 0$$

represents a straight line.

Take any point $(\alpha_0, \beta_0, \gamma_0)$ as pole, and let the equations of the radius vector from it to any point (α, β, γ) on the locus of the equation

$$l\alpha + m\beta + n\gamma = 0 \qquad (6)$$

be
$$\frac{\alpha - \alpha_0}{\lambda} = \frac{\beta - \beta_0}{\mu} = \frac{\gamma - \gamma_0}{\nu} = r. \qquad (7)$$

We have from (7)

$$\alpha = \alpha_0 + \lambda r,$$
$$\beta = \beta_0 + \mu r,$$
$$\gamma = \gamma_0 + \nu r;$$

and, substituting these values of α, β, γ in (6), we get

$$(l\lambda + m\mu + n\nu)r + (l\alpha_0 + m\beta_0 + n\gamma_0) = 0,$$

which can give only one value of r for each separate value of $(\alpha_0, \beta_0, \gamma_0)$, or of (λ, μ, ν).

Hence, no straight line can meet the locus in more than one point; the locus of (6) is therefore a straight line[a].

13. *Polar form of the general equation of the second degree.*

The method of the preceding Article may be applied to the general equation of the *second* degree,

$$\phi(\alpha, \beta, \gamma) = A\alpha^2 + B\beta^2 + C\gamma^2 + 2D\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta = 0.$$

---

[a] It will be observed, moreover, that if $(\alpha_0, \beta_0, \gamma_0)$ be taken *on* the locus, since in that case $l\alpha_0 + m\beta_0 + n\gamma_0 = 0$, r will always $= 0$, except when $l\lambda + m\mu + n\nu = 0$ also, (which, as will shortly be seen, is the condition that the radius vector should coincide in direction with the locus), and then it is indeterminate.

For putting, as before,

$$\alpha = \alpha_0 + \lambda r,$$
$$\beta = \beta_0 + \mu r,$$
$$\gamma = \gamma_0 + \nu r,$$

we get, by Taylor's Theorem,

$$\phi(\alpha_0, \beta_0, \gamma_0) + \left\{ \left(\frac{d\varphi}{d\alpha_0}\right)\lambda + \left(\frac{d\phi}{d\beta_0}\right)\mu + \left(\frac{d\phi}{d\gamma_0}\right)\nu \right\} r$$
$$+ \frac{1}{1.2}\left\{ \left(\frac{d^2\phi}{d\alpha_0^2}\right)\lambda^2 + \left(\frac{d^2\phi}{d\beta_0^2}\right)\mu^2 + \left(\frac{d^2\phi}{d\gamma_0^2}\right)\nu^2 + 2\left(\frac{d^2\phi}{d\beta_0\, d\gamma_0}\right)\mu\nu \right.$$
$$\left. + 2\left(\frac{d^2\phi}{d\gamma_0\, d\alpha_0}\right)\nu\lambda + 2\left(\frac{d^2\phi}{d\alpha_0\, d\beta_0}\right)\lambda\mu \right\} r^2 = 0,$$

or, since

$$\left(\frac{d^2\phi}{d\alpha_0^2}\right) = 2A,\ \left(\frac{d^2\phi}{d\beta_0^2}\right) = 2B,\ \left(\frac{d^2\phi}{d\gamma_0^2}\right) = 2C,$$
$$\left(\frac{d^2\phi}{d\beta_0\, d\gamma_0}\right) = 2D,\ \left(\frac{d^2\phi}{d\gamma_0\, d\alpha_0}\right) = 2E,\ \left(\frac{d^2\phi}{d\gamma_0\, d\alpha_0}\right) = 2F,$$

$$\phi(\alpha_0, \beta_0, \gamma_0) + \left\{ \left(\frac{d\phi}{d\alpha_0}\right)\lambda + \left(\frac{d\phi}{d\beta_0}\right)\mu + \left(\frac{d\phi}{d\gamma_0}\right)\nu \right\} r + \phi(\lambda, \mu, \nu) r^2 = 0, \tag{8}$$

which is the polar form of the general equation of the second degree; $(\alpha_0, \beta_0, \gamma_0)$ being the pole, and λ, μ, ν, the direction-cosines of the radius vector.

14. As it will be necessary sometimes to distinguish between the two forms of the equation of the straight line which have been investigated in Arts. 8—11, we propose to call the former the *symmetrical* and the latter the *homogeneous* form. The latter form will be denoted by $(l, m, n) = 0$, or simply by (l, m, n); the former as has been already stated, by (λ, μ, ν) or $(\cos a, \cos b, \cos c)$. Examples will be found at the end of the work which will familiarize the reader with the method of forming the symmetrical equations of a right line in particular cases.

15. *To construct geometrically the straight line whose equation is given in the homogeneous form.*

Suppose the given equation to be

$$l\alpha + m\beta + n\gamma = 0,$$

and, in order to find the co-ordinates of the points P, Q, R, (see fig. Art. 7) in which the locus intersects the sides of the triangle of reference, make $\alpha = 0$, $\beta = 0$, $\gamma = 0$, successively. We get (Prelim. chap. (A).), for P, Q and R respectively

$$\frac{\alpha}{0} = \frac{\beta}{n} = \frac{\gamma}{-m} = \frac{2S}{bn-cm},$$

$$\frac{\alpha}{-n} = \frac{\beta}{0} = \frac{\gamma}{l} = \frac{2S}{cl-an},$$

$$\frac{\alpha}{m} = \frac{\beta}{-l} = \frac{\gamma}{0} = \frac{2S}{am-bl}.$$

Hence these points may be obtained geometrically by drawing parallels to the sides of the triangle of reference at distances determined by the preceding equations. And, when any two of the three points P, Q, R, are found, the position of the straight line is known.

16. *If a straight line pass through the intersection of two other straight lines whose equations are*

$$l_1\alpha + m_1\beta + n_1\gamma = 0,$$
$$l_2\alpha + m_2\beta + n_2\gamma = 0,$$

its equation will be of the form

$$(l_1\alpha + m_1\beta + n_1\gamma) - k(l_2\alpha + m_2\beta + n_2\gamma) = 0,$$

k being an arbitrary constant.

For the equation $(l_1, m_1, n_1) - k(l_2, m_2, n_2) = 0$ (Art 12) must represent *some* straight line; also, it is satisfied by those values of α, β, γ, which satisfy the equations $(l_1, m_1, n_1) = 0$, $(l_2, m_2, n_2) = 0$, simultaneously. It therefore represents a straight line which passes through the point of intersection of the lines (l_1, m_1, n_1), (l_2, m_2, n_2).

17. *If the equations of three straight lines (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) be such that the sum of the left-hand members, when multiplied each by some constant, vanishes identically, these three straight lines meet in a point.*

For suppose L_1, L_2, L_3 to be certain multipliers, such that

$$L_1(l_1\alpha+m_1\beta+n_1\gamma)+L_2(l_2\alpha+m_2\beta+n_2\gamma)+L_3(l_3\alpha+m_3\beta+n_3\gamma) = 0.$$

Then it is manifest that the co-ordinates of the intersection of any two of the three straight lines (since they satisfy simultaneously the equations of that pair of straight lines,) will, by virtue of this relation, also satisfy the equation of the third.

Hence any one of the three given straight lines passes through the intersection of the other two.

18. In the following examples the given triangle is taken as the triangle of reference, so that the equations of its sides are $\alpha = 0$, $\beta = 0$, $\gamma = 0$.

19. *To find the equations of the bisectors of the angles of a triangle, and to show that they meet in a point.*

The straight line which bisects the $\angle A$ of the triangle ABC may be regarded as the locus of points which are equidistant from the two sides CA ($\beta = 0$), $AB(\gamma = 0)$; its equation, therefore, will be $\beta = \gamma$, or

$$\beta - \gamma = 0.$$

Similarly,

$$\gamma - \alpha = 0, \qquad (9)$$

$$\alpha - \beta = 0,$$

are the equations of the bisectors of the angles B and C respectively.

Also, since the left-hand members of (9), when added together, are identically equal to zero, the condition of Art. 17 is satisfied, and the three bisectors meet in a point.

20. If $(\alpha_0, \beta_0, \gamma_0)$ be the co-ordinates of this point, we shall have, by equations (9)

$$\alpha_0 = \beta_0 = \gamma_0 = \frac{2S}{a+b+c}, \text{ (Prelim. chap. } (A).),$$

$$= \frac{S}{s}, \qquad (10)$$

$2s$ being the perimeter of the triangle.

21. By Art. 16, the equation of a straight line which passes through the vertex A of the triangle of reference must be of the form

$$\beta + k_1\gamma = 0;$$

for it passes through the intersection of the lines whose equations are $\beta = 0$, $\gamma = 0$. This is otherwise evident, since its equation must be satisfied by the co-ordinates of A, $\left(\frac{2S}{a}, 0, 0\right)$.

Similarly, straight lines through B and C will be represented by equations of the forms

$$\gamma + k_2\alpha = 0,$$

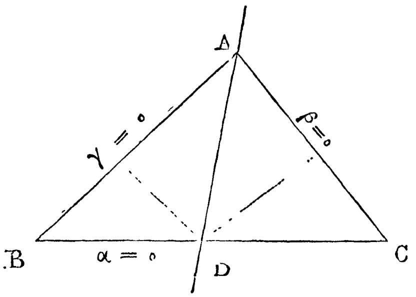
$$\alpha + k_3\beta = 0,$$

respectively.

22. *To find the equations of the bisectors of sides of the triangle ABC, and to shew that they meet in a point.*

Let D be the middle point of BC. The equation of AD, since

Fig. 5.



it passes through the intersection of $\beta = 0$, $\gamma = 0$, must (Art. 21) be of the form

$$\beta - k\gamma = 0. \tag{11}$$

Also, since D, $\left(0, \frac{a}{2}\sin C, \frac{a}{2}\sin B\right)$ is a point on the line, we have from (11)

$$\sin C - k \sin B = 0,$$

and therefore $k = \frac{\sin C}{\sin B}$, and, substituting this value for k in (11), we get for the equation of AD

$$\sin B\,\beta - \sin C\,\gamma = 0.$$

Similarly, we find $\sin C\,\gamma - \sin A\,\alpha = 0,$ (12)

and $\sin A\,\alpha - \sin B\,\beta = 0,$

for those of the other bisectors.

Also, since the sum of the left-hand members of these equations $= 0$ identically, the three bisectors of the sides of a triangle meet in a point. (Art. 17.)

23. This point, which we shall denote by $(\alpha_0, \beta_0, \gamma_0)$, is evidently such that its co-ordinates satisfy the relations

$$\sin A\alpha_0 = \sin B\beta_0 = \sin C\gamma_0, \tag{13}$$

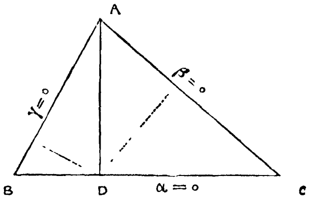
or $$a\alpha_0 = b\beta_0 = c\gamma_0;$$

whence, since $a\alpha_0 + b\beta_0 + c\gamma_0 = 2S$ (Art. 3),

$$a\alpha_0 = b\beta_0 = c\gamma_0 = \frac{2S}{3}. \tag{14}$$

24. *To find the equations of the perpendiculars let fall from the vertices of the triangle ABC upon the opposite sides, and to shew that they meet in a point.*

Fig. 6.



Let p_α represent the length of AD, the perpendicular from A on BC. The equation of AD must (Art. 21) be of the form

$$\beta - k\gamma = 0; \tag{15}$$

and, since $D(0, p_\alpha \cos C, p_\alpha \cos B)$ is a point on the line, we have

$$\cos C - k\cos B = 0;$$

whence, substituting for k in (15), we get for the equation of AD

$$\cos B\,\beta - \cos C\,\gamma = 0.$$

Similarly,
$$\begin{aligned}\cos C\,\gamma - \cos A\,\alpha &= 0,\\ \cos A\,\alpha - \cos B\,\beta &= 0,\end{aligned} \tag{16}$$

will be found to be the equations of the other perpendiculars.

Also, since the left-hand members of these equations, when added together, vanish identically, it follows that the three perpendiculars from the vertices of any triangle upon its opposite sides meet in a point. (Art. 17.)

25. If $(\alpha_0, \beta_0, \gamma_0)$ be the co-ordinates of this point, we shall have, as in Art. 23,

$$\cos A\alpha_0 = \cos B\beta_0 = \cos C\gamma_0, \tag{17}$$

and therefore, (Prelim. chap. (A).)

$$\frac{\alpha_0}{\sec A} = \frac{\beta_0}{\sec B} = \frac{\gamma_0}{\sec C} = \frac{2S}{a \sec A + b \sec B + c \sec C}. \tag{18}$$

26. In the above instances the actual values of the co-ordinates $(\alpha_0, \beta_0, \gamma_0)$ have been deduced; but it is to be observed that these are rarely required in practice. For, as the trilinear equations, when not homogeneous, may (Art. 4) be easily rendered so, it is, in general, sufficient for us to know the proportional values of the co-ordinates of a point.

Thus,
$$(\operatorname{cosec} A, \operatorname{cosec} B, \operatorname{cosec} C),$$
$$(\sec A, \quad \sec B, \quad \sec C)$$

(see Eqq. 13, 17) may be used respectively for the co-ordinates of the point of intersection of the bisectors of sides, and of that of the perpendiculars from the vertices upon the sides which subtend them.

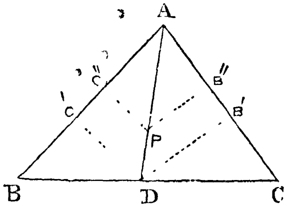
So again,
$$(1, \quad 1, \quad 1)$$
$$(\cos A, \cos B, \cos C)$$

may be taken instead of the co-ordinates of the inscribed and circumscribed circles respectively.

27. The equations of Arts. 22 and 24 are perhaps more easily obtained by the following geometrical method.

For the *bisectors of sides*:—Let $P(\alpha, \beta, \gamma)$ be *any* point on the bisector AD.

Fig. 7.



Draw DB', DB'' perpendicular to CA, and DC', DC'' perpendicular to AB.

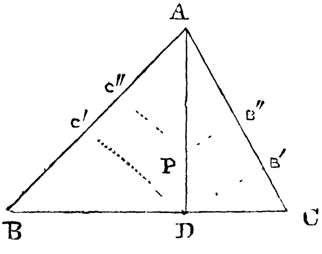
Then, by similar triangles,

$$\frac{\beta}{\gamma} = \frac{PB''}{PC''} = \frac{DB'}{DC'} = \frac{DC \sin C}{DB \sin B} = \frac{\sin C}{\sin B};$$

whence $$\sin B\,\beta - \sin C\,\gamma = 0; \text{ etc.}$$

For the *perpendiculars from the angular points*:—Let $P(\alpha, \beta, \gamma)$

Fig. 8.



be *any* point on the perpendicular AD, and let the perpendiculars DB', DC', PB', PB'' be drawn as before. We have

$$\frac{\beta}{\gamma} = \frac{PB''}{PC''} = \frac{DB'}{DC'} = \frac{AD \cos C}{AD \cos B} = \frac{\cos C}{\cos B};$$

and $$\cos B\,\beta - \cos C\,\gamma = 0; \text{ etc.}$$

CHAPTER II.

THE STRAIGHT LINE CONTINUED. RELATIONS BETWEEN THE CONSTANTS.

28. IN the foregoing chapter (Arts. 9—12) we have arrived at two forms of the equation of a right line which have been named respectively the *symmetrical* and the *homogeneous*. We proceed now to investigate certain important relations which connect the constants in these two forms with each other and with the elements of the triangle of reference.

29. First, however, that needless repetition may be avoided, let it be remarked, once for all, that the equations

$$\frac{\alpha-\alpha_1}{\cos a_1}=\frac{\beta-\beta_1}{\cos b_1}=\frac{\gamma-\gamma_1}{\cos c_1}=r \qquad (19)$$

and

$$l_1\alpha+m_1\beta+n_1\gamma=0, \qquad (20)$$

or, more briefly, $(\cos a_1, \cos b_1, \cos c_1)$ and (l_1, m_1, n_1), will invariably be employed to represent the *same* straight line. When the investigation requires the introduction of another straight line, this second line will be represented by $(\cos a_2, \cos b_2, \cos c_2)$ or (l_2, m_2, n_2), or by both, according as it may be convenient to use either or both of the two forms of the equation. Hence, wherever a straight line is represented by $(l_1, m_1, n_1)=0$, it is assumed that its direction-cosines are $\cos a_1$, $\cos b_1$, $\cos c_1$, and so conversely.

30. *The relations*

$$\left.\begin{aligned}\sin(b_1-c_1)&=\sin A,\\ \sin(c_1-a_1)&=\sin B,\\ \sin(a_1-b_1)&=\sin C\end{aligned}\right\} \quad (21) \qquad \left.\begin{aligned}\cos(b_1-c_1)&=-\cos A,\\ \cos(c_1-a_1)&=-\cos B,\\ \cos(a_1-b_1)&=-\cos C\end{aligned}\right\} \quad (22)$$

between the direction-angles of any right line and the angles of the triangle of reference, follow at once from the equations

$$\begin{aligned} 180^\circ - A &= b_1 - c_1, \\ 180^\circ - B &= c_1 - a_1, \\ 180^\circ + C &= -(a_1 - b_1), \end{aligned} \tag{23}$$

the truth of which appears from an inspection of the figure given in Art. 9.

31. *To prove the relations*

$$a \cos a_1 + b \cos b_1 + c \cos c_1 = 0, \tag{24}$$

$$a \sin a_1 + b \sin b_1 + c \sin c_1 = 0, \tag{25}$$

$$l_1 \cos a_1 + m_1 \cos b_1 + n_1 \cos c_1 = 0. \tag{26}$$

Equations (24) and (25) are easily obtained by projecting the triangle of reference upon the line $(\cos a_1, \cos b_1, \cos c_1)$, and upon any other straight line cutting it at right angles. (See Art. 34).

To prove the last, take the equations (19) and (20), (Art. 29). From the first of these we have

$$r = \frac{\alpha - \alpha_1}{\cos a_1} = \frac{\beta - \beta_1}{\cos b_1} = \frac{\gamma - \gamma_1}{\cos c_1}$$

$$= \frac{(l_1\alpha + m_1\beta + n_1\gamma) - (l_1\alpha_1 + m_1\beta_1 + n_1\gamma_1)}{l_1 \cos a_1 + m_1 \cos b_1 + n_1 \cos c_1}, \text{ (Prelim. chap (A).)}$$

But the numerator of the right-hand member vanishes by reason of (20). Hence we must have also

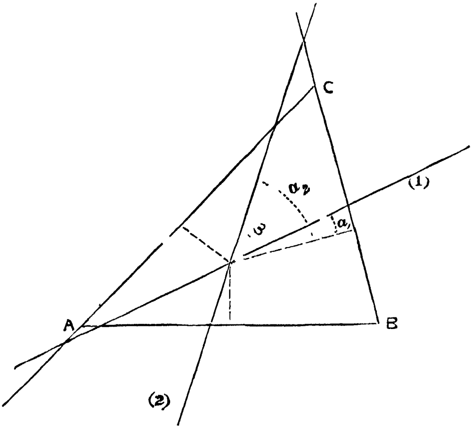
$$l_1 \cos a_1 + m_1 \cos b_1 + n_1 \cos c_1 = 0;$$

a relation which always holds between the direction-cosines of a right line and the co-efficients of its homogeneous equation.

32. *To find the equation of the straight line which passes through the point* $(\alpha_1, \beta_1, \gamma_1)$ *and makes an angle* ω *with the straight line* $(\cos a_1, \cos b_1, \cos c_1)$.

Let (1) in the figure represent the given straight line, whose

Fig. 9.



direction-angles are a_1, b_1, c_1; and let (2) be the straight line whose equation is required.

Suppose the direction-angles of (2) to be a_2, b_2, c_2; then, since it passes through the given point, its equation will be

$$\frac{\alpha-\alpha_1}{\cos a_2}=\frac{\beta-\beta_1}{\cos b_2}=\frac{\gamma-\gamma_1}{\cos c_2}=r. \tag{27}$$

Draw the co-ordinates of the point of intersection of (1) and (2); then, from the geometry, we have

$$a_2=\omega+a_1.$$

Similarly $$b_2=\omega+b_1, \tag{28}$$

and $$c_2=\omega+c_1.$$

Hence (27) becomes

$$\frac{\alpha-\alpha_1}{\cos(\omega+a_1)}=\frac{\beta-\beta_1}{\cos(\omega+b_1)}=\frac{\gamma-\gamma_1}{\cos(\omega+c_1)}=r, \tag{29}$$

which is the required equation.

33. If $\omega=\frac{\pi}{2}$, the straight lines (1) and (2) intersect each other

at right angles; and we have the following theorem, of which frequent use will hereafter be made.

If two straight lines cut each other at right angles in the point (a_1, β_1, γ_1), *their equations are of the form*

$$\frac{a-a_1}{\cos a_1}=\frac{\beta-\beta_1}{\cos b_1}=\frac{\gamma-\gamma_1}{\cos c_1}=r, \tag{30}$$

$$\frac{a-a_1}{\sin a_1}=\frac{\beta-\beta_1}{\sin b_1}=\frac{\gamma-\gamma_1}{\sin c_1}=r. \tag{31}$$

In other words, *the direction-cosines of two perpendicular right lines are of the form* (cos a_1 cos b_1, cos c_1), (sin a_1, sin b_1, sin c_1).

34. The result of the last Article affords us another proof of the relations (24) and (25) of Art. 31. For, remembering that

$$a\alpha+b\beta+c\gamma=2S \text{ always,} \qquad \text{(Art. 3. (1).)}$$

we have from equation (30)

$$r=\frac{a-a_1}{\cos a_1}=\frac{\beta-\beta_1}{\cos b_1}=\frac{\gamma-\gamma_1}{\cos c_1}$$

$$=\frac{(a\alpha+b\beta+c\gamma)-(a\alpha_1+b\beta_1+c\gamma_1)}{a\cos a_1+b\cos b_1+c\cos c_1} \quad \text{(Prelim. chap. (A).)}$$

$$=\frac{2S-2S}{a\cos a_1+b\cos b_1+c\cos c_1};$$

and, since the numerator vanishes, we must have always

$$a\cos a_1+b\cos b_1+c\cos c_1=0.$$

The relation

$$a\sin a_1+b\sin b_1+c\sin c_1=0$$

follows in a similar manner from equation (31) of the preceding Article.

34. *To prove the formulæ*

$$\begin{aligned}\sin^2 A&=\cos^2 b_1+\cos^2 c_1+2\cos b_1\cos c_1\cos A,\\ \sin^2 B&=\cos^2 c_1+\cos^2 a_1+2\cos c_1\cos a_1\cos B, \qquad (32)\\ \textit{and}\quad \sin^2 C&=\cos^2 a_1+\cos^2 b_1+2\cos a_1\cos b_1\cos C.\end{aligned}$$

It was shewn (Art. 30') that $\sin A = \sin(b_1 - c_1)$. We have therefore

$$\begin{aligned}\sin^2 A &= \sin^2(b_1 - c_1),\\ &= (1-\cos^2 b_1)\cos^2 c_1 + (1-\cos^2 c_1)\cos^2 b_1\\ &\qquad - 2\cos b_1 \cos c_1 \sin b_1 \sin c_1\\ &= \cos^2 b_1 + \cos^2 c_1 - 2\cos b_1 \cos c_1(-\cos A\\ &\qquad - \cos b_1 \cos c_1) - 2\cos^2 b_1 \cos^2 c_1\end{aligned}$$

Since (Art. 30), $\cos A = -\cos(b_1 - c_1)$;

$$\therefore \qquad \sin A = \cos^2 b_1 + \cos^2 c_1 + 2\cos b_1 \cos c_1 \cos A.$$

The others follow by symmetry.

35. *To prove the following symmetrical expressions for the area of the triangle of reference.*

$$\frac{4\,S^2}{abc} = a\cos A\cos^2 a_1 + b\cos B\cos^2 b_1 + c\cos C\cos^2 c_1, \qquad (33)$$

$$-\frac{4\,S^2}{abc} = a\cos b_1\cos c_1 + b\cos c_1\cos a_1 + c\cos a_1\cos b_1. \qquad (34)$$

By Art. 34 we have

$$\begin{aligned}4\,S^2 = b^2c^2\sin^2 A &= b^2c^2\{\cos^2 b_1 + \cos^2 c_1 + 2\cos b_1\cos c_1\cos A\}\\ &= b^2c^2(\cos^2 b_1 + \cos^2 c_1) + bc\cos A\,\{(b\cos b_1 + c\cos c_1)^2\\ &\qquad - b^2\cos^2 b_1 - c^2\cos^2 c_1\};\end{aligned}$$

or, since (Art. 31. (24).) $b\cos b_1 + c\cos c_1 = -a\cos a_1$,

$$\begin{aligned}4\,S^2 &= bc\cos A(a^2\cos^2 a_1) + c(c - b\cos A)(b^2\cos^2 b_1)\\ &\qquad + b(b - c\cos A)(c^2\cos^2 c_1)\\ &= abc\,\{a\cos A\cos^2 a_1 + b\cos B\cos^2 b_1 + c\cos C\cos^2 c_1\}.\end{aligned}$$

Again

$$4\,S^2 = b^2c^2\sin^2 A = b^2c^2\cos^2 b_1 + b^2c^2\cos^2 c_1 + 2b^2c^2\cos b_1\cos c_1\cos A;$$

or, since
$$b\cos b_1 = -(c\cos c_1 + a\cos a_1),$$
$$c\cos c_1 = -(a\cos a_1 + b\cos b_1),$$
and
$$2bc\cos A = b^2 + c^2 - a^2,$$
$$4\,S^2 = -bc^2\cos b_1(c\cos c_1 + a\cos a_1) - b^2c\cos c_1(a\cos a_1 + b\cos b_1) + bc(b^2+c^2-a^2)\cos b_1\cos c_1$$
$$= -abc\{a\cos b_1\cos c_1 + b\cos c_1\cos a_1 + c\cos a_1\cos b_1\}.$$

36. The results of the last Article aie of use when it is required to find a symmetrical expression for the distance between two given points (Art. 44). They also give us the following *symmetrical expressions for the sines of the angles of the triangle of reference;*

$$\frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} = \frac{\sin^2 C}{c^2} = \frac{a\cos A\cos^2 a_1 + b\cos B\cos^2 b_1 + c\cos C\cos^2 c_1}{abc}. \quad (35)$$

$$\text{also,} \quad = -\frac{a\cos b_1\cos c_1 + b\cos c_1\cos a_1 + c\cos a_1\cos b_1}{abc}. \quad (36)$$

37. It is to be observed that, since (Art. 33) $\sin a_1$, $\sin b_1$, $\sin c_1$, are the direction-cosines of some line, viz. of some line perpendicular to ($\cos a_1$, $\cos b_1$, $\cos c_1$,) the results of the preceding Articles remain true, if for the cosines of the direction-angles we substitute the corresponding sines. For obvious reasons this remark does not apply to formulæ which involve l_1, m_1, n_1.

Hence, besides
$$l_1\cos a_1 + m_1\cos b_1 + n_1\cos c_1 = 0,$$
we have the following relations in which λ, μ, ν, may be replaced either by $\cos a_1$, $\cos b_1$, $\cos c_1$, or by the sines of the same angles;
$$a\lambda + b\mu + c\nu = 0,$$
$$\mu^2 + \nu^2 + 2\mu\nu\cos A = \sin^2 A,$$
$$\gamma^2 + \lambda^2 + 2\nu\lambda\cos B = \sin^2 B,$$
$$\lambda^2 + \mu^2 + 2\lambda\mu\cos C = \sin^2 C,$$

$$\frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} = \frac{\sin^2 C}{c^2} = \frac{4\,S^2}{a^2 b^2 c^2} = \frac{a\cos A\lambda^2 + b\cos B\mu^2 + c\cos C\nu^2}{abc}.$$

$$\text{also} \quad = -\frac{a\mu\nu + b\nu\lambda + c\lambda\mu}{abc}.$$

38. By adding the two expressions involved in

$$\frac{4\,S^2}{abc} = a\cos A\lambda^2 + b\cos B\mu^2 + c\cos C\nu^2,$$

since, $\sin^2 a_1 + \cos^2 a_1 = 1$, etc. = etc.; or those involved in

$$-\frac{4S^2}{abc} = a\mu\nu + b\nu\lambda + c\lambda\mu,$$

since (Art. 30. (22).), $\cos(b_1 - c_1) = -\cos A$, etc. = etc.; we get in each case the expression

$$\frac{8\,S^2}{abc} = a\cos A + b\cos B + c\cos C, \tag{37}$$

which is sometimes useful, and may assist the student in remembering the more important formulæ from which it is derived[a].

39. *To write down the homogeneous equation of a straight line whose symmetrical equations are given.*

Let the given equations be

$$\frac{a - a_0}{\lambda} = \frac{\beta - \beta_0}{\mu} = \frac{\gamma - \gamma_0}{\mu}, \tag{38}$$

and suppose

$$la + m\beta + n\gamma = 0 \tag{39}$$

to be the homogeneous form of the equation of the same straight line

Since (a_0, β_0, γ_0) is a point on the line, we shall have

$$la_0 + m\beta_0 + n\gamma_0 = 0;$$

---

[a] This formula is easily proved independently; for the right-hand member

$$= \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{2abc} = \frac{8s(s-a)(s-b)(s-c)}{abc} = \frac{8S^2}{abc}.$$

also, (Art. 31. (26).) $l\lambda + m\mu + n\nu = 0$;

whence,
$$\frac{l}{\begin{vmatrix}\beta_0, & \gamma_0\\ \mu, & \nu\end{vmatrix}} = \frac{m}{\begin{vmatrix}\gamma_0, & \alpha_0\\ \nu, & \lambda\end{vmatrix}} = \frac{n}{\begin{vmatrix}\alpha_0, & \beta_0\\ \lambda, & \mu\end{vmatrix}}, \tag{40}$$

and (39) becomes

$$\begin{vmatrix}\beta_0, & \gamma_0\\ \mu, & \nu\end{vmatrix}\alpha + \begin{vmatrix}\gamma_0, & \alpha_0\\ \nu, & \lambda\end{vmatrix}\beta + \begin{vmatrix}\alpha_0, & \beta_0\\ \lambda, & \mu\end{vmatrix}\gamma = 0. \tag{41}$$

40. *To find the direction-cosines of a straight line whose equation is given in the homogeneous form.*

Let
$$l_1\alpha + m_1\beta + n_1\gamma = 0$$
be the given equation: then, since (Art. 31. (24), (26).)
$$a\cos a_1 + b\cos b_1 + c\cos c_1 = 0$$
and
$$l_1\cos a_1 + m_1\cos b_1 + n_1\cos c_1 = 0,$$
we have, by cross-multiplication,
$$\frac{\cos a_1}{\begin{vmatrix}b, & c\\ m_1, & n_1\end{vmatrix}} = \frac{\cos b_1}{\begin{vmatrix}c, & a\\ n_1, & l_1\end{vmatrix}} = \frac{\cos c_1}{\begin{vmatrix}a_1, & b_1\\ l_1, & m_1\end{vmatrix}}$$
$$= \frac{a\cos b_1\cos c_1 + b\cos c_1\cos a_1 + c\cos a_1\cos b_1}{\sqrt{a\begin{vmatrix}c, & a\\ n_1, & l_1\end{vmatrix}\begin{vmatrix}a, & b\\ l_1, & m_1\end{vmatrix} + b\begin{vmatrix}a, & b\\ l_1, & m_1\end{vmatrix}\begin{vmatrix}b, & c\\ m_1, & n_1\end{vmatrix} + c\begin{vmatrix}b, & c\\ m_1, & n_1\end{vmatrix}\begin{vmatrix}c, & a\\ n_1, & l_1\end{vmatrix}}}$$
(Prelim. chap. (*B*).)
$$= \frac{2S}{abc\sqrt{(l_1^2 + m_1^2 + n_1^2 - 2m_1n_1\cos A - 2n_1l_1\cos B - 2l_1m_1\cos C)}}$$
(Art. 35.)
$$= \frac{2S}{abc\{l_1, m_1, n_1\}}, \tag{42}$$

if we denote by $\{l_1, m_1, n_1\}$ the function of l_1, m_1, n_1, for which it has been substituted[b].

---

[b] In the same way the expression
$$(\sqrt{(a^2 + b^2 + c^2 - 2bc\cos A - 2ca\cos B - 2ab\cos C)}).$$
will be represented by $\{a, b, c.\}$

41. The direction-cosines of (l_1, m_1, n_1) may obviously be written in the somewhat more convenient form

$$\frac{\cos a_1}{\begin{vmatrix}\sin B, & \sin C\\ m_1, & n_1\end{vmatrix}} = \frac{\cos b_1}{\begin{vmatrix}\sin C, & \sin A\\ n_1, & l_1\end{vmatrix}} = \frac{\cos c_1}{\begin{vmatrix}\sin A, & \sin B\\ l_1, & m_1\end{vmatrix}} = \frac{1}{\{l_1, m_1, n_1\}} \quad (43)$$

the same abbreviation being made.

42. *To find the sines of the direction-angles of a straight line, when its equation is given in the homogeneous form.*

By equation (43) of the last Article, we have

$$\sin^2 a_1 = \frac{l_1, m_1, n_1{}^2 - (n_1 \sin B - m_1 \sin C)^2}{\{l_1, m_1, n_1\}^2}$$

$$= \frac{1}{\{l_1, m_1, n_1\}^2}\left[l_1^2 + m_1^2\cos^2 C + n_1^2\cos^2 B - 2m_1n_1(\cos A - \sin B \sin C) - 2n_1l_1 \cos B - 2l_1m_1 \cos C\right]$$

$$= \frac{(l_1 - m_1 \cos C - n_1 \cos B)^2}{\{l_1, m_1, n_1\}^2};$$

Therefore $$\sin a_1 = \pm \frac{l_1 - m_1 \cos C - n_1 \cos B}{\{l_1, m_1, n_1\}},$$

the values of $\sin b_1$, $\sin c_1$ following by symmetry. Hence

$$\frac{\sin a_1}{l_1 - m_1 \cos C - n_1 \cos B} = \frac{\sin b_1}{m_1 - n_1 \cos A - l_1 \cos C} = \frac{\sin c_1}{n_1 - l_1 \cos B - m_1 \cos A}$$

$$= \frac{1}{\{l_1, m_1, n_1\}}. \quad (44)$$

43. *To shew that*

$$l_1 \sin a_1 + m_1 \sin b_1 + n_1 \sin c_1 = \{l_1, m_1, n_1\}.$$

Since $l_1 \cos a_1 + m_1 \cos b_1 + n_1 \cos c_1 = 0$ (Art. 31. (24).), we have

$$(l_1 \sin a_1 + m_1 \sin b_1 + n_1 \sin c_1)^2 = (l_1 \sin a_1 + m_1 \sin b_1 + n_1 \sin c_1)^2 + (l_1 \cos a_1 + m_1 \cos b_1 + n_1 \cos c_1)_2$$

$$
\begin{aligned}
&= l_1^2 + m_1^2 + n_1^2 + 2m_1n_1 \cos(b_1 - c_1) \\
&\qquad + 2n_1l_1 \cos(c_1 - a_1) + 2l_1m_1 \cos(a_1 - b_1) \\
&= l_1^2 + m_1^2 + n_1^2 - 2m_1n_1 \cos A - 2n_1l_1 \cos B \\
&\qquad - 2l_1m_1 \cos C \\
&= \{l_1, m_1, n_1{}^2\}, \text{ (Art. 40.)}
\end{aligned}
$$

Hence

$$l_1 \sin a_1 + m_1 \sin b_1 + n_1 \sin c_1 = \{l_1, m_1, n_1\} \qquad (45)$$

44. *To shew that*

$$l_1 \cos(\omega + a_1) + m_1 \cos(\omega + b_1) + n_1 \cos(\omega + c_1) = -\{l_1, m_1, n_1\} \sin \omega.$$

Expanding, we get

$$
\begin{aligned}
l_1 \cos(\omega + a_1) + m_1 \cos(\omega + b_1) + n_1 \cos(\omega + c_1) &= (l_1 \cos a_1 + m_1 \cos b_1 \\
&\qquad + n_1 \cos c_1) \cos \omega \\
&\quad - (l_1 \sin a_1 + m_1 \sin b_1 \\
&\qquad + n_1 \sin c_1) \sin \omega \\
&= -(l_1 \sin a_1 + m_1 \sin b_1 \\
&\qquad + n_1 \sin c_1) \sin \omega, \\
&\qquad \text{(by Art. 31. (24).)}, \\
&= \{l_1, m_1, n_1\} \sin \omega, \quad (46)
\end{aligned}
$$

by the result of the last Article.

CHAPTER III.

THE STRAIGHT LINE CONTINUED. LINE AT INFINITY.

45. *To find the distance between two points.*

Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ be the given points, and r the distance between them. Then, if λ, μ, ν are the direction-cosines of the joining line, we have (Art. 9.)

$$r = \frac{\alpha_1 - \alpha_2}{\lambda} = \frac{\beta_1 - \beta_2}{\mu} = \frac{\gamma_1 - \gamma_2}{\nu};$$

whence, (Prelim. chap. (*C*).),

$$r^2 = \frac{a \cos A(\alpha_1-\alpha_2)^2 + b\cos B(\beta_1-\beta_2)^2 + c\cos C(\gamma_1-\gamma_2)^2}{a\cos A\lambda^2 + b\cos B\mu^2 + c\cos C\nu^2}$$

$$= \frac{abc}{4S^2}\{a\cos A(\alpha_1-\alpha_2)^2 + b\cos B(\beta_1-\beta_2)^2 + c\cos C(\gamma_1-\gamma_2)^2\}; \quad (47)$$

or, (Prelim. chap. (*B*).)

$$r^2 = \frac{a(\beta_1-\beta_2)(\gamma_1-\gamma_2) + b(\gamma_1-\gamma_2)(\alpha_1-\alpha_2) + c(\alpha_1-\alpha_2)(\beta_1-\beta_2)}{a\mu\nu + b\nu\lambda + c\lambda\mu}$$

$$= -\frac{abc}{4S^2}\{a(\beta_1-\beta_2)(\gamma_1-\gamma_2) + b(\gamma_1-\gamma_2)(\alpha_1-\alpha_2) + c(\alpha_1-\alpha_2)(\beta_1-\beta_2)\}; \quad (48)$$

(47) and (48) following by reason of equations (33) and (34), (Art. 35).

46. It is sometimes more convenient to write the results of the last Article in the following form:

$$r^2 = \frac{(\alpha_1-\alpha_2)^2\sin 2A + (\beta_1-\beta_2)^2\sin 2B + (\gamma_1-\gamma_2)^2\sin 2C}{2\sin A\sin B\sin C}, \quad (49)$$

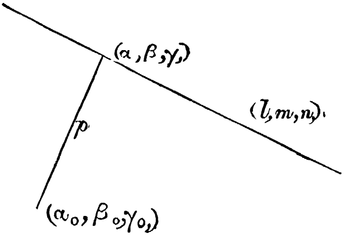
and

$$r^2 = -\frac{(\beta_1-\beta_2)(\gamma_1-\gamma_2)\sin A + (\gamma_1-\gamma_2)(\alpha_1-\alpha_2)\sin B + (\alpha_1-\alpha_2)(\beta_1-\beta_2)\sin C}{\sin A\sin B\sin C}. \quad (50)$$

47. *To find the length of the perpendicular from the point* $(\alpha_0, \beta_0, \gamma_0)$ *upon the straight line* (l_1, m_1, n_1).

The direction-cosines of the perpendicular will be (Art. 33) $\sin a_1$, $\sin b_1$, $\sin c_1$.

Fig. 10.



If therefore p be the required length, and α, β, γ the co-ordinates of the foot of the perpendicular, we shall have

$$\frac{\alpha-\alpha_0}{\sin a_1} = \frac{\beta-\beta_0}{\sin b_1} = \frac{\gamma-\gamma_0}{\sin c_1} = p\,;$$

whence

$$\alpha = \alpha_0 \pm \sin a_1 p$$
$$\beta = \beta_0 \pm \sin b_1 p$$
$$\gamma = \gamma_0 \pm \sin c_1 p$$

and, substituting these values for α, β, γ, in the equation

$$l_1\alpha + m_1\beta + n_1\gamma = 0,$$

we get

$$l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0 \pm (l_1 \sin a_1 + m_1 \sin b_1 + n_1 \sin c_1)p = 0,$$

and therefore

$$p = \pm \frac{l_1\alpha_0 + m_1\beta_1 + n_1\gamma_0}{l_1 \sin a_1 + m_1 \sin b_1 + n_1 \sin c_1}$$

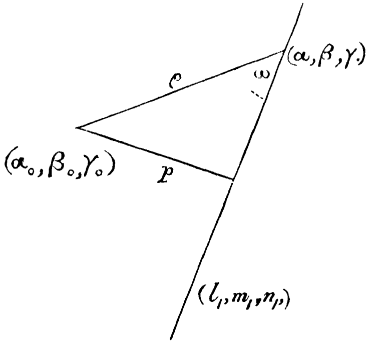
$$= \pm \frac{l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0}{\{l_1, m_1, n_1\}}, \text{ (Art. 43,)} \qquad (51)$$

where, as before,

$$\{l_1, m_1, n_1\} = \surd(l_1^2 + m_1^2 + n_1^2 - 2m_1n_1 \cos A - 2n_1l_1 \cos B - 2l_1m_1 \cos C).$$

48. Hence also, if a straight line be drawn from $(\alpha_0, \beta_0, \gamma_0)$ to meet the line (l_1, m_1, n_1) so as to make an angle ω with it, its length ρ is known; since, from the geometry of the figure, $\rho = \frac{p}{\sin \omega}$.

Fig. 11.



Thus

$$\rho = \pm \frac{l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0}{\{l_1, m_1, n_1\} \sin \omega}. \tag{52}$$

49. But this result may be arrived at independently; for the equation of the line which meets (l_1, m_1, n_1) at the given angle will be (Art. 32)

$$\frac{\alpha-\alpha_0}{\cos(\omega + a_1)} = \frac{\beta-\beta_0}{\cos(\omega + b_1)} = \frac{\gamma-\gamma_0}{\cos(\omega + c_1)} = \pm \rho,$$

and, proceeding as in Art. 46, we shall get finally

$$\rho = \pm \frac{l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0}{l_1 \cos(\omega+a_1)+m_1 \cos(\omega+b_1)+n_1 \cos(\omega+c_1)}$$

$$= \pm \frac{l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0}{\{l_1, m_1, n_1\} \sin \omega}, \text{ by Art. 44.}$$

50. *To find the angle included between the two straight lines* $(\cos a_1, \cos b_1, \cos c_1)$, (l_2, m_2, n_2).

Let ω be the angle between the two given straight lines.

Then, the direction-angles of the first being a_1, b_1, c_1, those of the second will be $(\omega + a_1)$, $(\omega + b_1)$, $(\omega + c_1)$, (Art. 32. (28).).

Hence, (Art. 31. (26).), we have the relation

$$l_2 \cos(\omega + a_1) + m_2 \cos(\omega + b_1) + n_2 \cos(\omega + c_1) = 0\,;$$

wherefore

$$l_2 \cos a_1 + m_2 \cos b_1 + n_2 \cos c_1 = \tan\omega (l_2 \sin a_1 + m_2 \sin b_1 + n_2 \sin c_1),$$

and
$$\tan\omega = \frac{l_2 \cos a_1 + m_2 \cos b_1 + n_2 \cos c_1}{l_2 \sin a_1 + m_2 \sin b_1 + n_2 \sin c_1}. \qquad (53)$$

We have also the following values for sin ω and cos ω:

$$\sin\omega = \frac{l_2 \cos a_1 + m_2 \cos b_1 + n_2 \cos c_1}{\sqrt{\{(l_2 \cos a_1 + m_2 \cos b_1 + n_2 \cos c_1)^2 + (l_2 \sin a_1 + m_2 \sin b_1 + n_2 \sin c_1)^2\}}}$$

$$= \frac{l_2 \cos a_1 + m_2 \cos b_1 + n_2 \cos c_1}{\{l_2, m_2, n_2\}}, \qquad (54)$$

$$\cos\omega = \frac{l_2 \sin a_1 + m_2 \sin b_1 + n_2 \sin c_1}{\{l_2, m_2, n_2\}}. \qquad (55)$$

51. *To determine the angle included between the two straight lines* (l_1, m_1, n_1), (l_2, m_2, n_2).

Using the results of the last Article, and remembering (Arts. 41, 42) that

$$\frac{\cos a_1}{\begin{vmatrix} \sin B, & \sin C \\ m_1, & n_1 \end{vmatrix}} = \frac{\cos b_1}{\begin{vmatrix} \sin C, & \sin A \\ n_1, & l_1 \end{vmatrix}} = \frac{\cos c_1}{\begin{vmatrix} \sin A, & \sin B \\ l_1, & m_1 \end{vmatrix}} = \frac{1}{\{l_1, m_1, n_1\}}$$

and

$$\frac{\sin a_1}{l_1 - m_1 \cos C - n_1 \cos B} = \frac{\sin b_1}{m_1 - n_1 \cos A - l_1 \cos C} = \frac{\sin c_1}{n_1 - l_1 \cos B - m_1 \cos A}$$

$$= \frac{1}{\{l_1, m_1, n_1\}},$$

we have for the tangent of the included angle

$$\frac{l_2 \begin{vmatrix} \sin B, & \sin C \\ m_1, & n_1 \end{vmatrix} + m_2 \begin{vmatrix} \sin C, & \sin A \\ n_1, & l_1 \end{vmatrix} + n_2 \begin{vmatrix} \sin A, & \sin B \\ l_1, & m_1 \end{vmatrix}}{l_1 l_2 + m_1 m_2 + n_1 n_2 - (m_1 n_2 + m_2 n_1) \cos A - (n_1 l_2 + n_2 l_1) \cos B - (l_1 m_2 + l_2 m_1) \cos C}$$

and the following values for tan ω, sin ω, cos ω, respectively,

$$\frac{\begin{vmatrix} m_1, n_1 \\ m_2, n_2 \end{vmatrix} \sin A + \begin{vmatrix} n_1, l_1 \\ n_2, l_2 \end{vmatrix} \sin B + \begin{vmatrix} l_1, m_1 \\ l_2, m_2 \end{vmatrix} \sin C}{l_1 l_2 + m_1 m_2 + n_1 n_2 - (m_1 n_2 + m_2 n_1)\cos A - (n_1 l_2 + n_2 l_1)\cos B - (l_1 m_2 + l_2 m_1)\cos C} \qquad (56)$$

$$\frac{\begin{vmatrix} m_1, n_1 \\ m_2, n_2 \end{vmatrix} \sin A + \begin{vmatrix} n_1, l_1 \\ n_2, l_2 \end{vmatrix} \sin B + \begin{vmatrix} l_1, m_1 \\ l_2, m_2 \end{vmatrix} \sin C}{\{l_1, m_1, n_1\}.\{l_2, m_2, n_2\}} \qquad (57)$$

$$\frac{l_1 l_2 + m_1 m_2 + n_1 n_2 - (m_1 n_2 + m_2 n_1)\cos A - (n_1 l_2 + n_2 l_1)\cos B - (l_1 m_2 + l_2 m_1)\cos C}{\{l_1, m_1, n_1\}.\{l_2, m_2, n_2\}} \qquad (58)$$

52. *To investigate the meaning of the equation*

$$a\alpha + b\beta + c\gamma = 0.$$

The given equation, being linear, must (Art. 12) represent *some* straight line.

Also the equation

$$(a\lambda + b\mu + c\nu)r + (a\alpha_0 + b\beta_0 + c\gamma_0) = 0$$

gives, as in Art. 12, the distance of the required locus from a point $(\alpha_0, \beta_0, \gamma_0)$ not on the locus, in any direction (λ, μ, ν).

Therefore, since (Art. 31. (24).)

$$a\lambda + b\mu + c\nu = 0$$

always, the radius vector is in *every* direction infinite.

Hence the equation

$$a\alpha + b\beta + c\gamma = 0,$$

or $$\sin A\alpha + \sin B\beta + \sin C\gamma = 0,$$

or, (Art. 3. (1).), $$\text{a constant} = 0,$$

must be interpreted as representing a straight line which lies altogether at an infinite distance.

53. It will be observed that, as $\{a, b, c\} = 0$ identically, the direction-cosines of this line are (Art. 41) indeterminate.

CHAPTER IV.

PROBLEMS ON THE STRAIGHT LINE.

54. *To find the co-ordinates of the point of intersection of the straight lines* (l_1, m_1, n_1), (l_2, m_2, n_2).

Let $(\alpha_0, \beta_0, \gamma_0)$ be the co-ordinates of the point of intersection of the given straight lines, then we shall have

$$l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0 = 0,$$

and

$$l_2\alpha_0 + m_2\beta_0 + n_2\gamma_0 = 0;$$

whence, by cross-multiplication,

$$\frac{\alpha_0}{\begin{vmatrix} m_1, & n_1 \\ m_2, & n_2 \end{vmatrix}} = \frac{\beta_0}{\begin{vmatrix} n_1, & l_1 \\ n_2, & l_2 \end{vmatrix}} = \frac{\gamma_0}{\begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix}} = \frac{2S}{a\begin{vmatrix} m_1, & n_1 \\ m_2, & n_2 \end{vmatrix} + b\begin{vmatrix} n_1, & l_1 \\ n_2, & l_2 \end{vmatrix} + c\begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix}} \tag{59}$$

55. *To find the condition that the three straight lines* (l_1, m_1, n_1) (l_2, m_2, n_2), (l_3, m_3, n_3) *may meet in a point.*

Suppose the point $(\alpha_0, \beta_0, \gamma_0)$ to be common to the three given straight lines. Then we must have simultaneously

$$l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0 = 0,$$

$$l_2\alpha_0 + m_2\beta_0 + n_2\gamma_0 = 0,$$

$$l_3\alpha_0 + m_3\beta_0 + n_3\gamma_0 = 0;$$

and the condition that these equations should co-exist is

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = 0. \tag{60}$$

56. *To find the condition that the three points* $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ *may lie in the same straight line.*

Suppose the three given points to lie on the right line whose equation is

$$l\alpha + m\beta + n\gamma = 0:$$

we shall then have

$$l\alpha_1 + m\beta_1 + n\gamma_1 = 0,$$

$$l\alpha_2 + m\beta_2 + n\gamma_2 = 0,$$

and $$l\alpha_3 + m\beta_3 + n\gamma_3 = 0:$$

and these equations cannot exist together, unless

$$\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix} = 0, \qquad (61)$$

which is therefore the required condition.

57. *To find the symmetrical equations of the straight line which passes through the two given points* $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$.

The equations of a straight line through $(\alpha_2, \beta_2, \gamma_2)$ must be of the form

$$\frac{\alpha - \alpha_2}{\lambda} = \frac{\beta - \beta_2}{\mu} = \frac{\gamma - \gamma_2}{\nu}. \qquad (62)$$

Also since $(\alpha_1, \beta_1, \gamma_1)$ is a point upon the line, we have

$$\frac{\alpha_1 - \alpha_2}{\lambda} = \frac{\beta_1 - \beta_2}{\mu} = \frac{\gamma_1 - \gamma_2}{\nu}. \qquad (63)$$

Hence, eliminating λ, μ, ν between (62) and (63) we get

$$\frac{\alpha - \alpha_2}{\alpha_1 - \alpha_2} = \frac{\beta - \beta_2}{\beta_1 - \beta_2} = \frac{\gamma - \gamma_2}{\gamma_1 - \gamma_2}, \qquad (64)$$

which are the equations required.

58. *To find the homogeneous equation of the straight line which passes through the two given points* $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$.

Suppose the equation of the joining line to be

$$l\alpha + m\beta + n\gamma = 0: \tag{65}$$

then, since the given points lie on the line, we shall have, to determine l, m, n,

$$l\alpha_1 + m\beta_1 + n\gamma_1 = 0, \tag{66}$$

and

$$l\alpha_2 + m\beta_2 + n\gamma_2 = 0. \tag{67}$$

From these, by cross-multiplication, we get

$$\frac{l}{\begin{vmatrix}\beta_1, & \gamma_1\\ \beta_2, & \gamma_2\end{vmatrix}} = \frac{m}{\begin{vmatrix}\gamma_1, & \alpha_1\\ \gamma_2, & \alpha_2\end{vmatrix}} = \frac{n}{\begin{vmatrix}\alpha_1, & \beta_1\\ \alpha_2, & \beta_2\end{vmatrix}},$$

whence, substituting in (65), we have for the required equation

$$\begin{vmatrix}\beta_1, & \gamma_1\\ \beta_2, & \gamma_2\end{vmatrix}\alpha + \begin{vmatrix}\gamma_1, & \alpha_1\\ \gamma_2, & \alpha_2\end{vmatrix}\beta + \begin{vmatrix}\alpha_1, & \beta_1\\ \alpha_2, & \beta_2\end{vmatrix}\gamma = 0; \tag{68}$$

or, more briefly,

$$\begin{vmatrix}\alpha, & \beta, & \gamma\\ \alpha_1, & \beta_1, & \gamma_1\\ \alpha_2, & \beta_2, & \gamma_2\end{vmatrix} = 0. \tag{69}$$

In the latter form the equation might have been written down at once as the eliminant of the three equations (65), (66) and (67).

59. *To find the homogeneous equation of the straight line which passes through the point* $(\alpha_0, \beta_0, \gamma_0)$, *and whose direction-cosines are* λ, μ, ν.

Let the equation be

$$l\alpha + m\beta + n\gamma = 0;$$

then, since $(\alpha_0, \beta_0, \gamma_0)$ is a point on the line,

$$l\alpha_0 + m\beta_0 + n\gamma_0 = 0:$$

also, (Art. 31. (26).), $l\lambda + m\mu + n\nu = 0$.

And, eliminating l, m, n, between these three equations, we have for the required equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_0, & \beta_0, & \gamma_0 \\ \lambda, & \mu, & \nu \end{vmatrix} = 0. \tag{70}$$

This equation may also be written thus:

$$\begin{vmatrix} \beta_0, & \gamma_0 \\ \mu, & \nu \end{vmatrix} \alpha + \begin{vmatrix} \gamma_0, & \alpha_0 \\ \nu, & \lambda \end{vmatrix} \beta + \begin{vmatrix} \alpha_0, & \beta_0 \\ \lambda, & \mu \end{vmatrix} \gamma = 0. \tag{71}$$

60. If the equations of the two preceding Articles be compared, it will be seen that the homogeneous equation of a right line may be formed with equal ease, whether the co-ordinates of two points on the line, or the co-ordinates of one such point and the direction-cosines of the line, be given.

61. *To find the equation of the straight line which joins the point* $(\alpha_0, \beta_0, \gamma_0)$ *to the point of intersection of the straight lines* (l_1, m_1, n_1), (l_2, m_2, n_2).

The required equation must (Art. 16) be of the form

$$l_1\alpha + m_1\beta + n_1\gamma + k(l_2\alpha + m_2\beta + n_2\gamma) = 0, \tag{72}$$

because it passes through the intersection of the given straight lines.

Also, since it passes through $(\alpha_0, \beta_0, \gamma_0)$, we have, to determine k,

$$l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0 + k(l_2\alpha_0 + m_2\beta_0 + n_2\gamma_0) = 0;$$

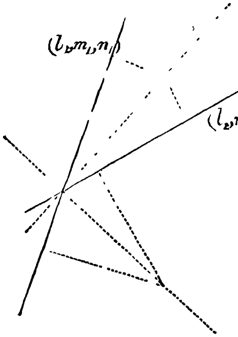
and (72) becomes, when this value is substituted for k,

$$(l_2\alpha_0 + m_2\beta_0 + n_2\gamma_0)(l_1\alpha + m_1\beta + n_1\gamma) = (l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0)(l_2\alpha + m_2\beta + n_2\gamma). \tag{73}$$

62. *To find the equation of the straight line which passes through the point of intersection of the straight lines* (l_1, m_1, n_1), (l_2, m_2, n_2), *and bisects the angle between them.*

The bisector is the locus of points the perpendicular distances of

Fig. 12.



which from the two given straight lines are equal; its equation therefore (Art. 47. (51).) will be

$$\frac{l_1\alpha + m_1\beta + n_1\gamma}{\{l_1, m_1, n_1\}} = \pm \frac{l_2\alpha + m_2\beta + n_2\gamma}{\{l_2, m_2, n_2\}}; \qquad (74)$$

the upper sign belonging to the *internal*, and the lower to the *external* bisector.

63. If the direction-angles of the two lines be given, and $(\alpha_0, \beta_0, \gamma_0)$ be their point of intersection, the bisectors will be respectively represented (Art. 33) by the symmetrical equations

$$\frac{\alpha - \alpha_0}{\cos\frac{a_1 + a_2}{2}} = \frac{\beta - \beta_0}{\cos\frac{b_1 + b_2}{2}} = \frac{\gamma - \gamma_0}{\cos\frac{c_1 + c_2}{2}}, \qquad (75)$$

and
$$\frac{\alpha - \alpha_0}{\sin\frac{a_1 + a_2}{2}} = \frac{\beta - \beta_0}{\sin\frac{b_1 + b_2}{2}} = \frac{\gamma - \gamma_0}{\sin\frac{c_1 + c_2}{2}}. \qquad (76)$$

For the direction-angles of the internal bisector are the Arithmetic means between a_1 and a_2, b_1 and b_2 c_1 and c_2, respectively; and the external bisector cuts the former at right angles.

64. *To find the condition that two given straight lines* (l_1, m_1, n_1), (l_2, m_2, n_2) *should be parallel to each other.*

The two given lines must, if they are parallel, have the same direction-cosines; hence (Art. 31), we must have, not only

$$a\cos a_1 + b\cos b_1 + c\cos c_1 = 0,$$

and $$l_1\cos a_1 + m_1\cos b_1 + n_1\cos c_1 = 0,$$

but also $$l_2\cos a_1 + m_2\cos b_1 + n_2\cos c_1 = 0.$$

whence, eliminating $\cos a_1$, $\cos b_1$, $\cos c_1$, we get for the required condition

$$\begin{vmatrix} a, & b, & c \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix} = 0. \tag{77}$$

65. *To find the equation of the straight line which passes through the point* $(\alpha_0, \beta_0, \gamma_0)$, *and is parallel to the straight line* (l_1, m_1, n_1).

This straight line may be regarded as the locus of points the perpendicular distances of which from (l_1, m_1, n_1) are equal to that of $(\alpha_0, \beta_0, \gamma_0)$ from the same right line. This (Art. 47) will be expressed as follows:

$$\frac{l_1\alpha + m_1\beta + n_1\gamma}{\{l_1, m_1, n_1\}} = \frac{l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0}{\{l_1, m_1, n_1\}},$$

or $$l_1\alpha + m_1\beta + n_1\gamma = l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0. \tag{78}$$

66. Or we may proceed as follows:

Since the straight line passes through the point $(\alpha_0, \beta_0, \gamma_0)$ and is parallel to (l_1, m_1, n_1), its direction-cosines (Art. 40) are respectively proportional to

$$\begin{vmatrix} b, & c \\ m_1, & n_1 \end{vmatrix}, \quad \begin{vmatrix} c, & a \\ n_1, & l_1 \end{vmatrix}, \quad \begin{vmatrix} a, & b \\ l_1, & m_1 \end{vmatrix};$$

and its equation (Art. 59. (70).) is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_0, & \beta_0, & \gamma_0 \\ bn_1 - cm_1, & cl_1 - an_1, & am_1 - bl_1 \end{vmatrix} = 0. \tag{79}$$

This equation is in the homogeneous form, and, being identical with

$$(a\alpha_0+b\beta_0+c\gamma_0)(l_1\alpha+m_1\beta+n_1\gamma)=(l_1\alpha_0+m_1\beta_0+n_1\gamma_0)(a\alpha+b\beta+c\gamma), \quad (80)$$

is equivalent to (78) of the last Article, since

$$a\alpha_0+b\beta_0+c\gamma_0=a\alpha+b\beta+c\gamma, \quad \text{(Art. 3)}.$$

67. From (78) it appears that *the general equation of a straight line parallel to* (l_1, m_1, n_1) *is*

$$l_1\alpha + m_1\beta + n_1\gamma = k \text{ (a constant)}, \quad (81)$$

and, conversely, *two straight lines are parallel when their equations differ only by a constant term.*

Equation (81) becomes (Art. 4), when rendered homogeneous,

$$l_1\alpha + m_1\beta + n_1\gamma = \frac{k}{2S}(a\alpha + b\beta + c\gamma),$$

and is therefore of the form

$$l\alpha_1 + m\beta_1 + n\gamma_1 = k'(a\alpha + b\beta + c\gamma), \quad (82)$$

the meaning of which (Arts. 16. 52) is that *every straight line which is parallel to* (l_1, m_1, n_1) *passes through the intersection of* (l_1, m_1, n_1) *with the straight line at infinity.*

This follows also from the condition of parallelism (Art. 64. (77).), which expresses (Art. 55. (60).) that the given straight lines must intersect on the straight line at infinity. Indeed we might have assumed this property of parallel straight lines, and deduced from it the condition of parallelism by the method of Art. 55. For the convenience of the student this form of proof is given in the next Article.

68. *To find the condition that the straight lines* (l_1, m_1, n_1), (l_2, m_2, n_2) *may be parallel.*

If the given lines be parallel they will intersect on the right line at infinity. Let $(\alpha_0, \beta_0, \gamma_0)$ be their point of intersection; we shall have

$$a\alpha_0 + b\beta_0 + c\gamma_0 = 0,$$

$$l_1\alpha_0 + m_1\beta_0 + n_1\gamma_0 = 0,$$

and $$l_2\alpha_0 + m_2\beta_0 + n_2\gamma_0 = 0:$$

and, eliminating α_0, β_0, γ_0 from these equations, we get for the required condition

$$\begin{vmatrix} a, & b, & c \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix} = 0.$$

69. It was shewn in Art. 33 that *the straight lines* ($\cos a_1$, $\cos b_1$, $\cos c_1$), ($\cos a_2$, $\cos b_2$, $\cos c_2$) *will intersect at right angles, if*

$$\cos a_2 = \sin a_1,$$
$$\cos b_2 = \sin b_1,$$
$$\cos c_2 = \sin c_1.$$

Hence also, since (Art. 31)

$$l_2 \cos a_2 + m_2 \cos b_2 + n_2 \cos c_2 = 0,$$

the condition that (l_2, m_2, n_2) *should be perpendicular to* ($\cos a_1$, $\cos b_1$, $\cos c_1$) *is*

$$l_2 \sin a_1 + m_2 \sin b_1 + n_2 \sin c_1 = 0. \qquad (83)$$

70. *To find the condition that the straight line* ($\cos a_2$, $\cos b_2$, $\cos c_2$) *should be perpendicular to the straight line* (l_1, m_1, n_1).

By Art. 33, we must have $\cos a_2 = \sin a_1$, etc. $=$ etc.; also, (Art. 42. (44).),

$$\frac{\sin a_1}{l_1 - m_1 \cos C - n_1 \cos B} = \frac{\sin b_1}{m_1 - n_1 \cos A - l_1 \cos C} = \frac{\sin c_1}{n_1 - l_1 \cos B - m_1 \cos A}.$$

Therefore the required condition is

$$\frac{\cos a_2}{l_1 - m_1 \cos C - n_1 \cos B} = \frac{\cos b_2}{m_1 - n_1 \cos A - l_1 \cos C} = \frac{\cos c_2}{n_1 - l_1 \cos B - n_1 \cos A}. \qquad (84)$$

71. *To find the condition that the straight lines* (l_1, m_1, n_1), (l_2, m_2, n_2) *should be mutually perpendicular.*

We have seen (Art. 33) that if the given straight lines intersect at right angles, $\cos a_2 = \sin a_1$, etc. $=$ etc. Hence the relation

$$l_2 \cos a_2 + m_2 \cos b_2 + n_2 \cos c_2 = 0$$

of Art. 31 becomes

$$l_2 \sin a_1 + m_2 \sin b_1 + n_2 \sin c_1 = 0,$$

or, since (Art. 42. (44).)

$$\frac{\sin a_1}{l_1 - m_1 \cos C - n_1 \cos B} = \frac{\sin b_1}{m_1 - n_1 \cos A - l_1 \cos C} = \frac{\sin c_1}{n_1 - l_1 \cos B - m_1 \cos A},$$

$$l_2(l_1 - m_1 \cos C - n_1 \cos B) + m_2(m_1 - n_1 \cos A - l_1 \cos C) + n_2(n_1 - l_1 \cos B - m_1 \cos A) = 0,$$

that is,

$$l_1 l_2 + m_1 m_2 + n_1 n_2 - (m_1 n_2 + m_2 n_1) \cos A - (n_1 l_2 + n_2 l_1) \cos B - (l_1 m_2 + l_2 m_1) \cos C = 0, \tag{85}$$

which is the required condition, another and somewhat more independent proof of which is given in the next Article.

72. Let the equation of the pair of lines be

$$\phi(\alpha, \beta, \gamma) = (l_1 \alpha + m_1 \beta + n_1 \gamma)(l_2 \alpha + m_2 \beta + n_2 \gamma) = 0$$

$$= l_1 l_2 \alpha^2 + m_1 m_2 \beta^2 + n_1 n_2 \gamma^2 + (m_1 n_2 + m_2 n_1) \beta\gamma + (n_1 l_2 + n_2 l_1) \gamma\alpha + (l_1 m_2 + l_2 m_1) \alpha\beta = 0.$$

The polar form of this equation, any point $(\alpha_0, \beta_0, \gamma_0)$ being taken as pole, is (Art. 13. (8).)

$$\phi(\lambda, \mu, \nu) r^2 + \left\{ \left(\frac{d\phi}{d\alpha_0}\right) \lambda + \left(\frac{d\phi}{d\beta_0}\right) \mu + \left(\frac{d\phi}{d\gamma_0}\right) \nu \right\} r + \phi(\alpha_0, \beta_0, \gamma_0) = 0.$$

Hence the radius vector is infinite, and therefore parallel to $(l_1, m_1, n_1), (l_2, m_2, n_2)$, when the direction-cosines satisfy the equation

$$\phi(\lambda, \mu, \nu) = 0, \tag{86}$$

or

$$l_1 l_2 \lambda^2 + m_1 m_2 \mu^2 + n_1 n_2 \nu^2 + (m_1 n_2 + m_2 n_1) \mu\nu + (n_1 l_2 + n_2 l_1) \nu\lambda + (l_2 m_2 + l_1 m_1) \lambda\mu = 0.$$

But (Art. 33) the values λ, μ, ν, which correspond to a pair of perpendicular lines, must be of the form $(\cos a_1, \cos b_1, \cos c_1)$ and $(\sin a_1, \sin b_1, \sin c_1)$, respectively.

We have therefore, by (86), both

$$\phi(\cos a_1, \cos b_1, \cos c_1) = 0,$$

and $$\phi(\sin a_1, \sin b_1, \sin c_1) = 0:$$

adding these, and remembering that $\cos(b_1 - c_1) = -\cos A$, etc. = etc., (Art. 30. (22).), we get

$$l_1 l_2 + m_1 m_2 + n_1 n_2 - (m_1 n_2 + m_2 n_1)\cos A - (n_1 l_2 + n_2 l_1)\cos B - (l_1 m_2 + l_2 m_1)\cos C = 0; \tag{87}$$

which is the condition sought.

73. It may be remarked that, since

$$al + bm + cn - (bn + cm)\cos A - (cl + an)\cos B - (am + bl)\cos A = 0$$

always, any straight line (l, m, n) whatever is to be regarded as perpendicular to the straight line at infinity (Art. 52).

74. It is only necessary to remind the reader that the conditions of the immediately preceding Articles, as well as the condition for the parallelism of two given straight lines, follow at once as corollaries from the results of Arts. 50, 51, of the last chapter.

75. *To find the symmetrical equations of the straight line which passes through the point* $(\alpha_0, \beta_0, \gamma_0)$, *and is perpendicular to the straight line* (l_1, m_1, n_1).

The required equation is, (Art. 33,)

$$\frac{\alpha - \alpha_0}{\sin a_1} = \frac{\beta - \beta_0}{\sin b_1} = \frac{\gamma - \gamma_0}{\sin c_1};$$

that is, (Art. 42,)

$$\frac{\alpha - \alpha_0}{l_1 - m_1\cos C - n_1\cos B} = \frac{\beta - \beta_0}{m_1 - n_1\cos A - l_1\cos C} = \frac{\gamma - \gamma_0}{n_1 - l_1\cos B - m_1\cos A}. \tag{88}$$

76. *To find the homogeneous equation of the straight line which passes through the point* $(\alpha_0, \beta_0, \gamma_0)$, *and is perpendicular to the straight line* (l_1, m_1, n_1).

Suppose $$l_2\alpha + m_2\beta + n_2\gamma = 0 \qquad (89)$$

to be the equation of the perpendicular.

Since it passes through $(\alpha_0, \beta_0, \gamma_0)$, we have

$$l_2\alpha_0 + m_2\beta_0 + n_2\gamma_0 = 0. \qquad (90)$$

Also, since it is perpendicular to (l_1, m_1, n_1), we have (Art. 69, (83),)

$$l_2 \sin a_1 + m_2 \sin b_1 + n_2 \sin c_1 = 0. \qquad (91)$$

Hence, eliminating l_2, m_2, n_2, between equations (89), (90) and (91), we obtain for the required equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_0, & \beta_0, & \gamma_0 \\ \sin a_1, & \sin b_1, & \sin c_1 \end{vmatrix} = 0. \qquad (92)$$

Or, (Art. 42,)

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_0, & \beta_0, & \gamma_0 \\ l_1 - m_1\cos C - n_1\cos B, & m_1 - n_1\cos A - l_1\cos C, & n_1 - l_1\cos B - m_1\cos A \end{vmatrix} = 0. \qquad (93)$$

77. *To find the homogeneous equation of the straight line which passes through the point* $(\alpha_0, \beta_0, \gamma_0)$, *and makes an angle* ω *with the straight line* (l_1, m_1, n_1).

Assuming $$l_2\alpha + m_2\beta + n_2\gamma = 0, \qquad (94)$$

for the equation of this straight line, we have, since $(\alpha_0, \beta_0, \gamma_0)$ lies upon it,

$$l_2\alpha_0 + m_2\beta_0 + n_2\gamma_0 = 0. \qquad (95)$$

Also, since its direction-cosines (Art. 32) are $\cos(\omega + a_1)$, $\cos(\omega + b_1)$, $\cos(\omega + c_1)$, the relation (26) of Art. 31 becomes

$$l_2 \cos(\omega + a_1) + m_2 \cos(\omega + b_1) + n_2 \cos(\omega + c_1) = 0. \qquad (96)$$

Therefore, eliminating l_2, m_2, n_2 between (94), (95) and (96), we have for the required equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_0, & \beta_0, & \gamma_0 \\ \cos(\omega+a_1), & \cos(\omega+b_1), & \cos(\omega+c_1) \end{vmatrix} = 0. \qquad (97)$$

But (Arts. 41, 42) $\cos(\omega + a_1) = \cos\omega . \cos a_1 - \sin\omega . \sin a_1$

$$= \frac{1}{\{l_1, m_1, n_1\}} \left\{ \cos\omega \begin{vmatrix} \sin B, & \sin C \\ m_1, & n_1 \end{vmatrix} - \sin\omega (l_1 - m_1 \cos C - n_1 \cos B) \right\}$$

$$= \frac{n_1 \sin(\omega + B) + m_1 \sin(\omega - C) - l_1 \sin\omega}{\{l_1, m_1, n_1\}},$$

with similar values for $\cos(\omega + b_1)$, $\cos(\omega + c_1)$.

Hence (97) may be written

$$\begin{vmatrix} \alpha, & \alpha_0, & l_1 \sin\omega - m_1 \sin(\omega - C) - n_1 \sin(\omega + B) \\ \beta, & \beta_0, & m_1 \sin\omega - n_1 \sin(\omega - A) - l_1 \sin(\omega + C) \\ \gamma, & \gamma_0, & n_1 \sin\omega - l_1 \sin(\omega - B) - m_1 \sin(\omega + A) \end{vmatrix} = 0. \qquad (98)$$

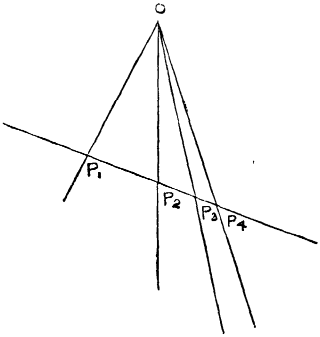
CHAPTER V.

ANHARMONIC RATIOS. HARMONIC POINTS AND PENCILS. HOMOGRAPHIC SYSTEMS. INVOLUTION.

78. In the present Chapter a short account will be given of the theory of Anharmonic and Harmonic section, so far as it relates to the properties of the point and line. For a more complete investigation of the subject the reader is referred to geometrical treatises [a].

79. When a *pencil of rays*, originating at a *centre* O, is cut by a *transversal* in points $P_1, P_2, P_3, \ldots$, $P_1P_2, P_1P_3, P_2P', \ldots$ are

Fig. 13.



conveniently employed to represent not only the corresponding segments of the transversal, but likewise the angles which those segments subtend at the centre of the pencil. In the same way shall occasionally use $P_1, P_2, P_3, \ldots$ to denote the rays OP_1, OP_2 $OP_3, \ldots$

With reference to the *range* of points $P_1, P_2, P_3, \ldots$ the transversal is sometimes called the *axis* of the system.

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[a] I may mention Townsend's "Modern Geometry," Hodges, Smith, and Co., Dublin, 1863—5.

80. *Definition of Anharmonic ratio.* Let the segment (or angle) P_1P_3 be divided, as in the figure of the last Article, by the two points (or lines) P_2, P_4; and let the two ratios $\frac{P_1P_2}{P_3P_2}$, $\frac{P_1P_4}{P_3P_4}$ $\left(\text{or } \frac{\sin P_1P_2}{\sin P_3P_2}, \frac{\sin P_1P_4}{\sin P_3P_4}\right)$ be denoted by ρ_2, ρ_4 (or σ_2, σ_4), respectively.

Then the ratio $\rho_2 : \rho_4$ (or $\sigma_2 : \sigma_4$) and its reciprocal $\rho_4 : \rho_2$ (or $\sigma_4 : \sigma_2$) are the two *anharmonic ratios* which are due to the division of the segment (or angle) P_1P_3.

Written at full length this pair of Anharmonic ratios will be

$$\frac{P_1P_2}{P_3P_2} : \frac{P_1P_4}{P_3P_4}, \text{ and its reciprocal,}$$

$$\left(\text{or } \frac{\sin P_1P_2}{\sin P_3P_2} : \frac{\sin P_1P_4}{\sin P_3P_4}, \text{ and its reciprocal}\right) :$$

but the following abbreviations are often used; viz.

$$\{P_1P_3, P_2P_4\} \text{ and } \{P_1P_3, P_4P_2\}, \tag{99}$$

to express the pair of ratios for the system of points, and

$$\{O.P_1P_3, P_2P_4\} \text{ and } \{O.P_1P_3, P_4P_2\} \tag{100}$$

to represent the corresponding ratios for the pencil.

81. But when four points on a common axis (or rays through a common vertex), P_1, P_2, P_3, P_4, are given, any one out of the six segments (or angles) P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, P_3P_4, may be regarded as the divided segment (or angle), and we shall have accordingly twelve anharmonic ratios; viz. the following six,

$$\{P_1P_2, P_3P_4\}, \{P_1P_3, P_2P_4\}, \{P_1P_4, P_2P_3\}, \text{ etc.,}$$

and their six reciprocals

$$\{P_1P_2, P_4P_3\}, \{P_1P_3, P_4P_2\}, \{P_1P_4, P\,P_2\}, \text{ etc.:}$$

(or the corresponding ratios of sines). Their number, however, is only half of that stated above; for it will be found that the ratios of the segments (or sines of the segments) of P_1P_3, made by P_2P_4, are the same in sign and magnitude as those of the segments (or sines of the segments) of P_2P_4, made by P_1, P_3; so that $\{P_1P_3, P_2P_4\} = \{P_2P_4, P_1P_3\}$, and so on. Hence *for a system of four points (or lines) there are but six different Anharmonic ratios, whereof three are the reciprocals of the remaining three: viz.*

$$\{P_1P_2\ P_3P_4\},\ \{P_1P_3, P_2P_4\},\ \{P_1P_4, P_2P_3\},$$

$$\{P_1P_2, P_4P_3\},\ \{P_1P_3, P_4P_2\},\ \{P_1P_4, P_3P_2\}.$$

82. These six Anharmonic ratios may be represented briefly by X, Y, Z, and their reciprocals; and from an inspection of their actual values it appears that

$$XYZ = -1, \text{ and } \frac{1}{X}\cdot\frac{1}{Y}\cdot\frac{1}{Z} = -1. \qquad (101)$$

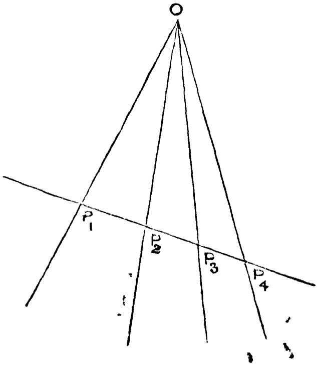
The student will have no difficulty in proving that they are also connected by the following relations;

$$X + \frac{1}{Y} = 1,\ \ Y + \frac{1}{Z} = 1,\ \ Z + \frac{1}{X} = 1. \qquad (102)$$

83. *If a pencil of four concurrent lines be cut by a transversal, the anharmonic ratios of the four points of intersection are equal to the corresponding ratios of the pencil, and are therefore the same for any position of the transversal.*

Let a pencil, having its centre at O, be cut by a transversal in the points P_1, P_2, P_3, P_4.

Fig. 14.



Using the notation of Art. 80, we have

$$\{P_1P_3,\ P_2P_4\} = \frac{P_1P_2}{P_3P_2} : \frac{P_1P_4}{P_3P_4}$$

$$= \frac{\Delta OP_1P_2}{\Delta OP_3P_2} : \frac{\Delta OP_1P_4}{\Delta OP_3P_4}, \text{ (Euc. vi. 1,)}$$

$$= \frac{OP_1 . OP_2 . \sin P_1P_2}{OP_1 . OP_3 . \sin P_3P_2} : \frac{OP_1 . OP_4 . \sin P_1P_4}{OP_3 . OP_4 . \sin P_3P_4}$$

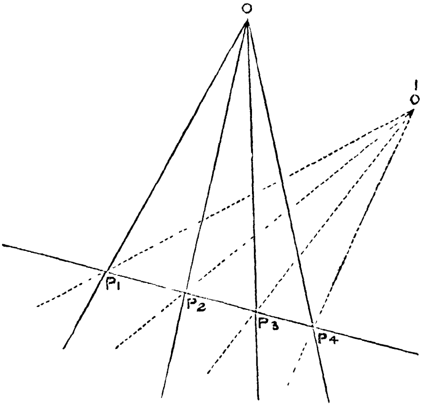
$$= \frac{\sin P_1P_2}{\sin P_3P_2} : \frac{\sin P_1P_4}{\sin P_3P_4}$$

$$= \{O.P_1P_3,\ P_2P_4\}. \tag{103}$$

And the same may be proved with respect to the remaining five anharmonic ratios of the range.

84. It follows from the theorem of the preceding Article that *if two pencils originating at O and O′, are such that their rays intersect, two and two, in points which lie in the same straight line, they will have the same anharmonic ratios.*

Fig. 15.



For, if P_1, P_2, P_3, P_4 be the points of intersection of the two pencils, then, since their anharmonic ratios are equal to the corresponding ratios of each of the pencils, we have

$$\{O.P_1P_3, P_2P_4\} = \{P_1P_3, P_2P_4\}$$
$$= \{O'.P_1P_3, P_2P_4\} \qquad (104)$$
$$\text{etc.} = \text{etc.},$$

and the pencils are therefore *equi-anharmonic*[b].

85. Any two pencils which are so mutually related that their corresponding rays intersect in pairs collinearly are said to be in *perspective*, or in *homology* with each other; and the line on which their common points lie is called the *axis of perspective*, or the *axis of homology*.

86. *Definition of harmonic section.* Any segment (or angle) is said to be *harmonically* divided by two points (or lines) when each of its two anharmonic ratios (Art. 80) $= -1$. Thus the line (or angle) $P_1 P_3$ will be divided harmonically by the points (or lines) P_2, P_4, provided that $\frac{P_1P_2}{P_3P_2} : \frac{P_1P_4}{P_3P_4} \left(\text{or } \frac{\sin P_1P_2}{\sin P_3P_2} : \frac{\sin P_1P_4}{\sin P_3P_4}\right)$, or its reciprocal, $= -1$; and then the points (or lines) P_2, P_4, are said to be *harmonic conjugates* with respect to P_1, P_3; and P_1, P_2, P_3, P_4 form a *harmonic range* (or *pencil*).

87. The *equation of harmonicism*

$$\frac{P_1P_2}{P_3P_2} : \frac{P_1P_4}{P_3P_4} = -1 \qquad (105)$$

may be written in the form

$$\frac{P_1P_2}{P_3P_2} = -\frac{P_1P_4}{P_3P_4}. \qquad (106)$$

Hence we conclude that *a segment which is harmonically divided is divided internally and externally in the same ratio.* This is sometimes taken as a definition of harmonic section.

---

[b] Equi-anharmonic systems of points or lines are said to be *homographic*. The two pencils in fig. 15 are both *homographic* and *perspective*, (Art. 85).

88. Another definition is afforded by the following property, which is easily deduced from (105). *If the segment P_1, P_3 be harmonically divided by P_2, P_4, and C be the point which bisects P_1P_3; then*

$$CP_1^2 = CP_3^2 = CP_2 . CP_4. \qquad (107)$$

89. The condition of harmonic section may also be written in the form

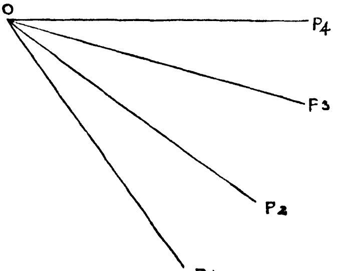
$$\{P_1P_3, P_2P_4\} = \{P_1P_3, P_4P_2\}, \qquad (108)$$

$$(\text{or} \quad \{O.P_1P_3, P_2P_4\} = \{O.P_1P_3, P_4P_2\}); \qquad (109)$$

for the only case in which the two reciprocal ratios of a divided segment (or angle) are equal to each other is that in which the dividing points (or rays) are harmonic conjugates of the two points (or rays) by which the divided segment (or angle) is terminated[c].

90. *To find the anharmonic ratios of a pencil of four concurrent lines whose direction-angles are given.*

Fig. 16.



Let the direction-angles of the four rays OP_1, OP_2, OP_3, OP_4, (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), (a_4, b_4, c_4) respectively. Then if

For (108) gives $\left(\frac{P_1P_2}{P_3P_2} : \frac{P_1P_4}{P_3P_4}\right)^2 = 1$. Therefore $\frac{P_1P_2}{P_3P_2} : \frac{P_1P_4}{P_3P_4} = \pm 1$; and, neglecting the positive sign, since it makes the points P_2, P_4, coincident, we have $\frac{P_1P_2}{P_3P_2} : \frac{P_1P_4}{P_3P_4} = -1$, the condition of harmonic section, (Art. 87. (105).).

$(\alpha_0, \beta_0, \gamma_0)$ be the co-ordinates of O, the equations of the rays will be

$$\frac{\alpha-\alpha_0}{\cos a_1} = \frac{\beta-\beta_0}{\cos \beta_1} = \frac{\gamma-\gamma_0}{\cos c_1}, \text{ etc.}$$

Also, as in Art. 32,

$$\angle P_1P_2 = a_1-a_2 = b_1-b_2 = c_1-c_2,$$

with similar values for the other angles of the figure.

Hence, to take only one of the six anharmonic ratios (Art. 81), viz.

$$\frac{\sin P_1P_2}{\sin P_3P_2} : \frac{\sin P_1P_4}{\sin P_3P_4},$$

we have for its equivalent in terms of the direction-angles

$$\frac{\sin(a_1-a_2)}{\sin(a_3-a_2)} : \frac{\sin(a_1-a_4)}{\sin(a_3-a_4)}, \tag{110}$$

or either of the expressions derivable from this by the substitution of (b_1, b_2, b_3) and (c_1, c_2, c_3) successively for (a_1, a_2, a_3).

The remaining five ratios may be formed in the same manner.

91. *To find the condition that four concurrent straight lines, whose direction-angles are given, should form a harmonic pencil.*

Recurring to the figure and notation of the last Article, we have, as the equivalent of the relation

$$\frac{\sin P_1P_2}{\sin P_3P_2} : \frac{\sin P_1P_4}{\sin P_3P_4} = -1 \text{ (Art. 87)},$$

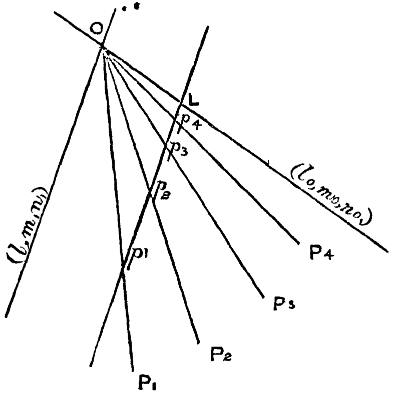
$$\frac{\sin(a_1-a_2)}{\sin(a_3-a_2)} : \frac{\sin(a_1-a_4)}{\sin(a_3-a_4)} = -1, \tag{111}$$

in which, as before, (a_1, a_2, a_3) may be replaced by (b_1, b_2, b_3) or by (c_1, c_2, c_3).

If the condition (111) be satisfied, the pencil will be harmonic, and $(\cos a_2, \cos b_2, \cos c_2)$ and $(\cos a_4, \cos b_4, \cos c_4)$ will be harmonic conjugates with respect to $(\cos a_1, \cos b_1, \cos c_1)$ and $(\cos a_3, \cos b_3, \cos c_3)$.

92. *To find the anharmonic ratios of a pencil of four right lines which pass through the intersection of the given lines* (l, m, n), (l_0, m_0, n_0).

Fig. 17.



Let the equations of the four rays OP_1, OP_2, OP_3, OP_4 (Art. 16) be

$$\begin{aligned}(l_0, m_0, n_0) - k_1(l, m, n) &= 0,\\ (l_0, m_0, n_0) - k_2(l, m, n) &= 0,\\ (l_0, m_0, n_0) - k_3(l, m, n) &= 0,\\ (l_0, m_0, n_0) - k_4(l, m, n) &= 0,\end{aligned} \qquad (112)$$

respectively.

Draw any transversal, parallel to (l, m, n), meeting the successive rays of the pencil in p_1, p_2, p_3, p_4, and (l_0, m_0, n_0) in L. Then, (Art. 83. (103).),

$$\begin{aligned}\{O.P_1P_3, P_2P_4\} &= \{p_1p_3, p_2p_4\}\\ &= \frac{p_1p_2}{p_3p_2} : \frac{p_1p_4}{p_3p_4}\\ &= \frac{p_1L-p_2L}{p_3L-p_2L} : \frac{p_1L-p_4L}{p_3L-p_4L}\\ &= \frac{k_1-k_2}{k_3-k_4} : \frac{k_1-k_4}{k_3-k_4}{}^{\text{d}},\end{aligned} \qquad (113)$$

and the other five ratios may be formed in a similar manner.

---

[d] For p_1L, p_2L, etc. are proportional to the lengths of the perpendiculars

93. From (113) it appears that the anharmonic ratios of the system of rays are independent of the constants in the equations of the given pair of lines (l_0, m_0, n_0), (l, m, n).

Hence, if there be another pencil of lines having for their equations

$$\begin{aligned}(l'_0, m'_0, n'_0) - k_1(l', m', n') &= 0,\\ (l'_0, m'_0, n'_0) - k_2(l', m', n') &= 0,\\ (l'_0, m'_0, n'_0) - k_3(l', m', n') &= 0,\\ (l'_0, m'_0, n'_0) - k_4(l', m', n') &= 0,\end{aligned} \qquad (114)$$

this pencil will have the same anharmonic ratios as that of the last Article; in other words, the equations (112) and (114) represent *homographic* systems.

94. *To find the condition that four straight lines which pass through the intersection of the pair of lines (l_0, m_0, n_0), (l, m, n) should form a harmonic pencil.*

Let us employ the same notation as in Art. 92. The equation of harmonicism

$$\frac{P_1P_2}{P_3P_2} : \frac{P_1P_4}{P_3P_4} = -1,$$

(Art. 87. (105).), becomes, by reason of (113),

$$\frac{k_1 - k_2}{k_3 - k_2} : \frac{k_1 - k_4}{k_3 - k_4} = -1; \qquad (115)$$

and, if this condition hold, the second and fourth lines of (114) will be harmonic conjugates with respect to the first and third.

---

from p_1, p_2, etc. upon (l_0, m_0, n_0), and therefore, as will now be shewn, to k_1, k_2, etc. respectively.

Suppose δ to be the length of the perpendicular upon (l, m, n) from any point on the transversal. Then (Art. 46) the perpendicular from p_1[(α, β, γ) say] upon $(l_0, m_0, n_0) = \dfrac{l_0\alpha + m_0\beta + n_0\gamma}{\{l_0, m_0, n_0\}} = \dfrac{l\alpha + m\beta + n\gamma}{\{l_0, m_0, n_0\}} k_1$ [(112) first eq.]

$$= \frac{\{l, m, n\}}{\{l_0, m_0, n_0\}} \cdot \frac{l\alpha + m\beta + n\gamma}{\{l, m, n\}} \cdot k_1 = \frac{\{l, m, n\}}{\{l_0, m_0, n_0\}} \cdot \delta k_1 \text{ (Art. 47)} = Ck_1,$$ where C is a constant for all points on the transversal.

95. *To find the form of the equations of any pair of straight lines which are harmonic conjugates with respect to the given pair* (l_0, m_0, n_0), (l, m, n).

Let
$$(l_0, m_0, n_0) - k_2(l, m, n) = 0 \tag{116}$$
$$(l_0, m_0, n_0) - k_4(l, m, n) = 0 \tag{117}$$
be the equations of the pair of conjugates.

Applying the condition (115) to the equations

$$\begin{aligned}(l_0, m_0, n_0) &= 0\\ (l_0, m_0, n_0) - k_2(l, m, n) &= 0\\ (l, m, n) &= 0\\ (l_0, m_0, n_0) - k_4(l, m, n) &= 0,\end{aligned}$$

we find that (116) and (117) will represent harmonic conjugates with respect to the given lines provided that

$$\left[\frac{k_1 - k_2}{k_3 - k_2} : \frac{k_1 - k_4}{k_3 - k_4}\right]_{\substack{k_1 = 0\\ k_3 = \infty}} = -1,$$

or
$$k_2 = -k_4 = \kappa \text{ (suppose)}.$$

Hence *four straight lines whose equations are of the form*

$$\left.\begin{aligned}(l_0, m_0, n_0) &= 0\\ (l, m, n) &= 0\end{aligned}\right\} \tag{118}$$

$$\left.\begin{aligned}(l_0, m_0, n_0) + \kappa(l, m, n) &= 0\\ (l_0, m_0, n_0) - \kappa(l, m, n) &= 0\end{aligned}\right\} \tag{119}$$

form a harmonic pencil, (118) *and* (119) *being conjugate pairs.*

96. Cor. Since the equations

$$\begin{aligned}(l_1, m_1, n_1) &= 0\\ (l_2, m_2, n_2) &= 0\end{aligned}$$

$$\frac{l_1\alpha + m_1\beta + n_1\gamma}{\{l_1, m_1, n_1\}} \pm \frac{l_2\alpha + m_2\beta + n_2\gamma}{\{l_2, m_2, n_2\}} = 0,$$

(Art. 62. (74).), are of the required form, we see that *the internal and external bisectors of the angle included by any pair of right lines are harmonic conjugates with respect to them.*

97. In the preceding Articles we have taken the most general case. It will be at once seen that the harmonic pencil formed by the lines $\beta = 0$, $\gamma = 0$, $\beta + \gamma = 0$, $\beta - \gamma = 0$, affords an instance of the theorem of Art. 96: also that the pencil

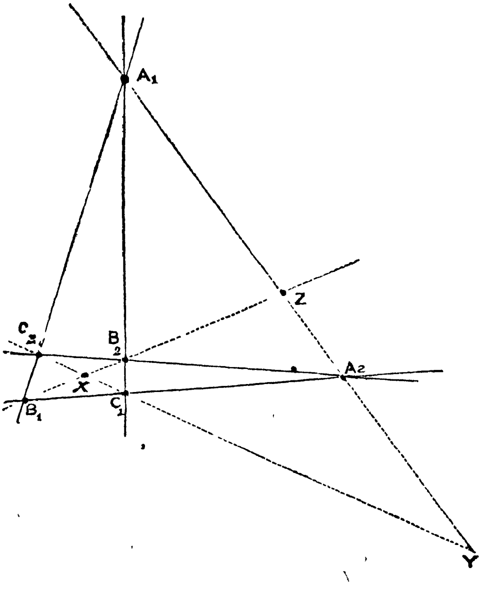
$$\begin{aligned} \beta - k_1\gamma &= 0 \\ \beta - k_2\gamma &= 0 \\ \beta - k_3\gamma &= 0 \\ \beta - k_4\gamma &= 0 \end{aligned} \qquad (120)$$

is homographic with (112) and (114).

98. *To prove the harmonic properties of a complete quadrilateral.*

Let B_1C_1, C_1B_2, B_2C_2, C_2B_1, be the sides of the *tetragram*; and

Fig. 18.



let A_1, A_2; B_1, B_2; C_1, C_2 be the pairs of intersections of opposite sides; also let X, Y, Z be the points of intersection of the three diagonals.

Take $A_1B_1C_1$ for the triangle of reference, and let the equations of its sides, taken in order, be $\alpha = 0$, $\beta = 0$, $\gamma = 0$, respectively.

Assume for the equations of the two diagonals XY, ZX

$$l\alpha - m\beta = 0 \ldots (XY)$$

$$n\gamma - l\alpha = 0 \ldots (ZX)$$

respectively. The equation of B_2C_2, the fourth side of the quadrilateral, will be of the form $n\gamma - l\alpha + k\beta = 0$; also of the form $l\alpha - m\beta + k'\gamma = 0$, (Art. 16): and, identifying these two equations, we get $k = m$, $k' = -n$; from either of which we find

$$l\alpha - m\beta - n\gamma = 0 \ldots (B_2C_2)$$

to be the equation required. Also, since the line A_1X passes through the intersections of $\beta = 0$ with $\gamma = 0$, and of $XY\,(l\alpha - m\beta = 0)$ with $ZX\,(n\gamma - l\alpha = 0)$, its equation will be of the form $\beta + k\gamma = 0$, or $l\alpha - m\beta + k'(n\gamma - l\alpha) = 0$; and identifying, as before, we get $k' = 1$, $k = -\frac{nk'}{m} = -\frac{n}{m}$; therefore the equation of A_1X is

$$m\beta - n\gamma = 0 \ldots (A_1X).$$

Again, since YZ passes through the intersections of $\beta = 0$ with $\gamma = 0$, and of $\alpha = 0$ with B_2C_2 $(l\alpha - m\beta - n\gamma = 0)$, we obtain, by a similar process, for its equation,

$$m\beta + n\gamma = 0 \ldots (YZ).$$

Hence
$$A_1C_1\,(\beta = 0),$$
$$A_1C_2\,(\nu = 0),$$
$$A_1X\,(m\beta - n\gamma = 0),$$
$$A_1Y\,(m\beta + n\gamma = 0),$$

(Art. 95,) form a harmonic pencil, and C_1, C_2, X, Y are four harmonic points.

In the same way it may be shewn that

$$\{B_1 . A_1A_2, YZ\}, \{C_1 . B_1B_2, XY\}$$

are harmonic pencils, and, consequently,

$$\{A_1A_2, YZ\}, \{B_1B_2, XY\}$$

harmonic ranges.

The triangle XYZ is sometimes called the *harmonic triangle*, since its sides are harmonically divided by the pairs of points A_1, A_2; B_1, B_2; C_1, C_2, respectively.

99. The reader will observe that the harmonic properties of the *quadrangle* (or *tetrastigm*) are also established by the proof just given in the case of the quadrilateral. We have, therefore, as the result of Art. 98, the two following reciprocal theorems.

> (a) *In every tetragram the three pairs of opposite intersections divide harmonically the three sides of the triangle determined by their three lines of connection.*
>
> (β) *In every tetrastigm the three pairs of opposite connectors divide harmonically the three angles of the triangle determined by their three points of intersection.* (Townsend's "Modern Geometry," Art. 236.)

HOMOGRAPHIC SYSTEMS OF POINTS AND LINES.

100. *Definition of homography.*—Two rows of points on any axes (or pencils of rays from any centres), $p_1, p_2, p_3, \ldots$ and $q_1, q_2, q_3, \ldots$, are said to be *homographic* when they correspond in such a manner that the anharmonic ratios of *any* four points (or rays) of the one are equal to those of their four correspondents in the other. (Art. 84, note.)

The homography of the two systems is expressed by the equation

$$\{p_1p_2p_3 \ldots\} = \{q_1q_2q_3 \ldots\}$$
$$(\text{or } \{O . p_1p_2p_3 \ldots\} = \{O . q_1q_2q_3 \ldots\}),$$

but in the following Articles, for the sake of brevity, the former notation only will be used; the student will have no difficulty in seeing where the latter is implied, and the ratios of segments may be replaced by the corresponding ratios of sines.

101. *Three pairs of corresponding points (or rays) are sufficient to determine two homographic systems;* for if we suppose the three pairs p_1, q_1; p_2, q_2; p_3, q_3, to be given, the correspondent q_4 of any fourth point (or ray) p_4 (taken in the first system) is known from the homographic relation

$$\{p_1 p_2 p_3 p_4\} = \{q_1 q_2 q_3 q_4\}. \tag{121}$$

102. *Point on either axis which corresponds to the point at infinity on the other.* Let I, J be those points of two homographic systems $\{p_1, p_2, p_3 \ldots\}$, $\{q_1, q_2, q_3 \ldots\}$, which correspond respectively to q_∞, p_∞, the points at infinity on the two axes.

We have, by the condition of homography (Art. 100),

$$\{p_1 p_2 p_3 I\} = \{q_1 q_2 q_3 q_\infty\},$$

that is,
$$\frac{p_1 p_3}{p_2 p_3} : \frac{p_1 I}{p_2 I} = \frac{q_1 q_3}{q_2 q_3} : \frac{q_1 q_\infty}{q_2 q_\infty},$$

$$= \frac{q_1 q_3}{q_2 q_3}. \tag{122}$$

Similarly,
$$\frac{q_1 q_3}{q_2 q_3} : \frac{q_1 J}{q_2 J} = \frac{p_1 p_3}{p_2 p_3}. \tag{123}$$

103. From (122) and (123) it appears that *if the correspondent of the point at infinity on either axis, and two pairs of corresponding points, be given, the correspondent of any third point, taken on either axis, is known.*

104. *If the points at infinity on the two axes correspond, the systems will be similar:* for then

$$\{p_1 p_2 p_3 p_\infty\} = \{q_1 q_2 q_3 q_\infty\},$$

whence
$$\frac{p_1 p_3}{p_2 p_3} = \frac{q_1 q_3}{q_2 q_3},$$

or,
$$\frac{p_3 p_1}{p_2 p_3} = \frac{q_3 q_1}{q_2 q_3}.$$

Similarly,
$$\frac{p_2 p_3}{p_3 p_1} = \frac{q_2 q_3}{q_3 q_1} \text{ and therefore } \frac{p_3 p_1}{p_1 p_2} = \frac{q_3 q_1}{q_1 q_2}.$$

Therefore
$$p_2 p_3 : p_3 p_1 : p_1 p_2 = q_2 q_3 : q_3 q_1 : q_1 q_2.$$

And, conversely, *the points at infinity on the axes of two similar homographic systems correspond.*

COAXAL (OR CONCENTRIC) HOMOGRAPHIC SYSTEMS.

105. *Double points and lines.—In every system of two homographic rows on the same axis (or pencils of rays from a common centre) there exist two points (or rays), called* double points (*or* rays), *which are their own correspondents.* For suppose the two coaxal systems to be $\{p_1 p_2 p_3 \ldots\}$, $\{q_1 q_2 q_3 \ldots\}$. Since they are homographic, we have

$$\{p_1 p_2 p_3 \ldots\} = \{q_1 q_2 q_3 \ldots\}. \tag{124}$$

Let I, J (Art. 102) be the points which correspond to the points at infinity p_∞, q_∞, and suppose x to be a point which belongs to both systems and is its own correspondent; we have by (124)

$$\{p_1 p_\infty I x\} = \{q_1 J q_\infty x\},$$

or,
$$\frac{p_1 I}{p_\infty I} : \frac{p_1 x}{p_\infty x} = \frac{q_1 q_\infty}{J q_\infty} : \frac{q_1 x}{J x};$$

therefore
$$\frac{p_1 x}{p_1 I} = \frac{q_1 x}{J x}$$

$$= \frac{p_1 x - p_1 q_1}{p_1 x - p_1 J}$$

(by introducing an origin p_1), whence we get

$$\overline{p_1 x}|^2 - (p_1 I + p_1 J)\overline{p_1 x}| + p_1 I . p_1 q_1 = 0, \tag{125}$$

a quadratic which determines two positions of x, its distance being measured from p_1.

If C be the middle point of the segment IJ, we have

$$p_1C = \frac{p_1I + p_1J}{2},$$

and (125) becomes

$$\overline{p_1x}|^2 - 2p_1C\,.\,\overline{p_1x}| + p_1I\,.\,p_1q_1 = 0. \tag{126}$$

There are therefore two double points (which we shall call P and Q), real or imaginary. They will manifestly be on opposite sides of the origin if $p_1C = 0$, that is, if the origin be at C the middle point of IJ; since their position will then (126) be given by the equation

$$\overline{Cx}|^2 + CI\,.\,CC' = 0, \tag{127}$$

in which C' is the correspondent of C.

106. Hence *the segments* IJ and PQ *are concentric.* Also from (127) it appears that if $CC' = 0$, P and Q coincide in the point C.

107. *Any pair of correspondents (say p_1, q_1) divide PQ into segments whose anharmonic ratios are constant.* For since, by their definition, P and Q are their own correspondents, we have (124), from the homography of the systems, p_2, q_2 being any other pair of correspondents,

$$\{p_1p_2PQ\} = \{q_1q_2PQ\},$$

or,

$$\frac{p_1P}{p_2P} : \frac{p_1Q}{p_2Q} = \frac{q_1P}{q_2P} : \frac{q_1Q}{q_2Q};$$

in other words, the ratio

$$\frac{p_1P}{q_1P} : \frac{p_1Q}{q_1Q}$$

is constant.

108. From the theorem of the last Article it follows that, *if one of the double points* (Q, suppose) *be at infinity*, the ratio $\frac{p_1P}{q_1P}$, for any

pair of correspondents p_1, q_1, is constant: that is to say, *the systems are similar.*

ON INVOLUTION.

109. *Definition of involution.—When two homographic coaxal rows (or concentric pencils) are such that every point (or ray) has the same correspondent, to whichever system it be regarded as belonging, the two rows (or pencils) are said to form a system in involution,* and the corresponding constituents are called *conjugates.*

110. *Two pairs of conjugates are sufficient to determine a system in involution.* For, if two points p_1, p_2, and their conjugates q_1, q_2, be given, the conjugate of any fifth point p_3, is known (121) from the equation

$$\{p_1q_1p_2q_3\} = \{q_1p_1q_2p_3\}. \tag{128}$$

111. It follows at once from (128) and the other similar equations that *any three pairs of conjugates of two homographic rows (or pencils) in involution are connected by the following relation,*

$$\frac{p_2q_1}{p_3q_1} \cdot \frac{p_3q_2}{p_1q_2} \cdot \frac{p_1q_3}{p_2q_3} = 1,$$

$$\left(or\ \frac{\sin p_2q_1}{\sin p_3q_1} \cdot \frac{\sin p_3q_2}{\sin p_1q_2} \cdot \frac{\sin p_1q_3}{\sin p_2q_3} = 1;\right) \tag{129}$$

and *conversely, when two homographic rows (or concentric pencils) are such that these relations hold between any three pairs of corresponding constituents, the rows (or pencils) are in involution.*

112. *Any pair of conjugates are harmonic conjugates with respect to the double points of the involution.* For, since P, Q (Art. 105) are self-conjugates, we have, supposing p_1, q_1 to be any pair of conjugate points,

$$\{p_1q_1PQ\} = \{q_1p_1PQ\},$$

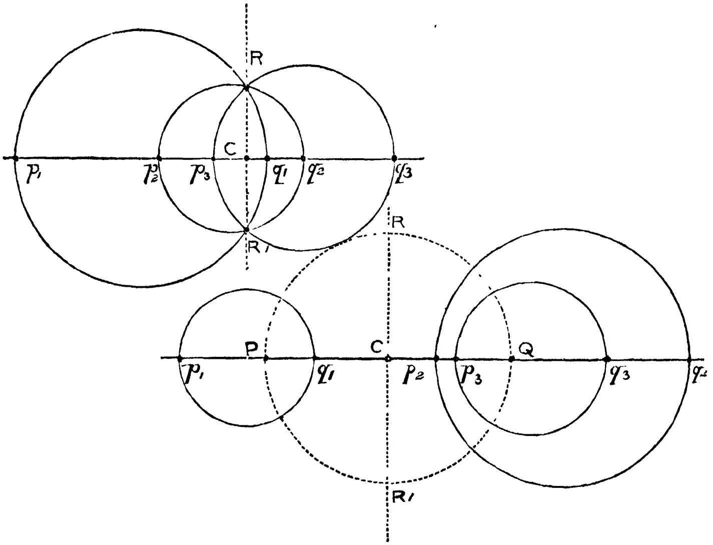
which shews (Art. 89. (108).) that p_1, q_1 are harmonic conjugates with respect to P and Q.

113. Hence (Art. 88. (107).), if C be the point of bisection of PQ, we have

$$CP^2 = CQ^2 = Cp_1 . Cq_1 = Cp_2 . Cq_2 = \ldots = \ldots .$$

C is called the *centre* of the involution, and evidently, if circles be described on the segments p_1q_1, p_2q_2, p_3q_3, . . . as diameters, they will

Fig. 19.



all pass through the same two points, and C will be that point on the line of centres through which the radical axis of the system passes.

114. The results of Arts. 112, 113, afford definitions of involution which are more convenient than the one given in Art. 109, since they admit of a simple geometrical interpretation by reference to the well-known properties (i.) of harmonic points, (ii.) of a system of circles having a common radical axis. Thus any one of the following definitions might have been given.

115. *When three or more pairs of collinear points (or concurrent*

lines) *are harmonic conjugates with respect to a fixed pair, they are said to form a row* (*or pencil*) *in involution, and the fixed pair are called the double points* (*or lines*) *of the system.*

116. Or, *when three or more pairs of collinear points*, p_1, q_1; p_2, q_2; p_3, q_3; . . ., *are so situated with reference to a fixed point C on their common axis that*

$$Cp_1 \cdot Cq_1 = Cp_2 \cdot Cq_2 = Cp_3 \cdot Cq_3 = \ldots$$
$$= a \text{ constant}$$
$$= \pm k^2 \text{ (say)}, \qquad (130)$$

the points are said to be in involution, and the point C is called the centre of the system, the involution being said to be *positive* or *negative* according as k^2 is affected by the upper or lower sign. The *double points P, Q,* since they are given by the equation

$$Cx^2 \pm k^2 = 0 \qquad (131)$$

(130), are equidistant from C and on opposite sides of it, and will be real or imaginary according as the involution is of the positive or negative kind.

117. Or again; *if a system of three or more pairs of collinear points* p_1, q_1; p_2, q_2; p_3, q_3; . . . *be such that the circles described about the segments* p_1q_1, p_2q_2, p_3q_3, . . . *as diameters have a common radical axis, the system is said to be in involution. The point C, in which the radical axis meets the axis of the collinear system, is called the* centre *of the involution, and the involution is said to be* positive *or* negative *according as the points* p_1, p_2, p_3, . . . *and their conjugates* q_1, q_2, q_3, . . . *lie on the same or on opposite sides of this central point. In the former case there are two points, P and Q,*—viz. the limiting points of the system of circles (being, in fact, circles with infinitesimal radii)—equidistant from C and on opposite sides of it, *which, when considered with reference to the collinear system, are their own conjugates, and are therefore called the* double points *of the involution. They are the points in which the axis of the involution is met by the circle, of radius k* (131), *which cuts the above system of circles orthogonally. In the latter case such double points do not exist,* there being no points

belonging to the system which possess the property of being self-conjugate.

In both cases the centre C is the conjugate of the point at infinity.

118. It will be observed that the definitions of Arts. 116, 117, do not apply to a system of *lines* in involution, which may be defined as in Arts. 109, 115. It is obvious, however, that *any* pencil of rays through an involution of points forms a system of lines in involution, of which the corresponding rays are those which pass through conjugate points. Also the rays which pass through the double points will be the double lines of the involution. Similarly, a pencil of lines in involution is cut by any transversal in points which are also in involution.

119. *In a pencil of lines in involution there exist always two conjugate rays which are at right angles to each other:* for suppose p_1, q_1; p_2, q_2; p_3, q_3, to be pairs of conjugates of the involution in which the pencil is cut by any transversal, and let circles be described on the segments p_1q_1, p_2q_2, p_3q_3; these circles (Art. 113) will have a common radical axis. Let now that circle of the system be described which passes through the vertex O of the pencil, and let pq be its intercept on the transversal: p, q are evidently conjugates of the involution of points, and Op, Oq, consequently, conjugate rays of the given pencil. Also Op, Oq are rectangular, since pq is a diameter.

120. *If more than one pair of conjugates of a pencil in involution be rectangular, every pair will be so;* for then the vertex of the pencil is evidently one of the two common points of the system of circles described as explained in the last Article.

121. From the definition given of involution in Art. 115 it appears that *straight lines whose equations are of the form*

$$\begin{aligned}(l_0, m_0, n_0) + k_1(l, m, n) &= 0\\(l_0, m_0, n_0) + k_2(l, m, n) &= 0\\(l_0, m_0, n_0) + k_3(l, m, n) &= 0\\\ldots\ldots\ldots &= 0\end{aligned} \qquad (132)$$

form a pencil which is in involution with the concentric homographic pencil (Art. 93) represented by the following equations,

$$\begin{aligned}(l_0, m_0, n_0) - k_1(l, m, n) &= 0,\\ (l_0, m_0, n_0) - k_2(l, m, n) &= 0,\\ (l_0, m_0, n_0) - k_3(l, m, n) &= 0,\\ \ldots\ldots\ldots &= 0;\end{aligned} \tag{133}$$

the straight lines

$$\begin{aligned}(l_0, m_0, n_0) &= 0\\ (l, m, n) &= 0\end{aligned} \tag{134}$$

being the double lines of the system. For we have seen (Art. 95) that (132) and (133), taken in pairs, are harmonic conjugates with respect to (134).

122. Similarly the straight lines

$$\begin{aligned}\beta \pm \gamma &= 0\\ \beta \pm k_1\gamma &= 0\\ \beta + k_2\gamma &= 0\\ \ldots\ldots &= 0\end{aligned} \tag{135}$$

form a system in involution of which

$$\begin{aligned}\beta &= 0\\ \gamma &= 0\end{aligned} \tag{136}$$

are the double lines, since each pair of (135) forms with (136) a harmonic pencil.

CHAPTER VI.

THE GENERAL EQUATION OF THE SECOND DEGREE.

123. We now proceed to discuss the trilinear equation of the second degree, the most general form of which (Art. 13) is

$$\phi(\alpha,\beta,\gamma) = A\alpha^2 + B\beta^2 + C\gamma^2 + 2D\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta = 0; \quad (137)$$

and whenever, in this chapter, the equation $\phi(\alpha, \beta, \gamma) = 0$, or the curve $\phi(\alpha, \beta, \gamma) = 0$, is spoken of, the reader will understand that the *complete* equation (137), or its locus, is intended. An investigation of various modified forms of the equation of the second degree will be found in the next chapter.

124. *To shew that the general equation of the second degree always represents a conic section.*

It was shewn (Art. 13) that the general equation of the second degree, $\phi(\alpha, \beta, \gamma) = 0$, may be thrown into the polar form

$$\phi(\lambda, \mu, \nu) r^2 + \left\{ \left(\frac{d\phi}{d\alpha_0}\right)\lambda + \left(\frac{d\phi}{d\beta_0}\right)\mu + \left(\frac{d\phi}{d\gamma_0}\right)\nu \right\} r + \phi(\alpha_0, \beta_0, \gamma_0) = 0. \quad (138)$$

Now, since $(\alpha_0, \beta_0, \gamma_0)$ may be *any* point and the direction-cosines (λ, μ, ν) may have any values whatever, this quadratic shews that if a straight line be drawn through a point $(\alpha_0, \beta_0, \gamma_0)$, in *any* direction, it will meet the locus of $\phi(\alpha, \beta, \gamma) = 0$ in two points, which are either real, coincident, or imaginary.

Hence the locus of the general equation is a curve of the second degree, that is to say, a conic section.

125. *To find the conditions that the general equation of the second degree should represent a Hyperbola, a Parabola, or an Ellipse.*

From the equation (138) it appears that the directions of the asymptotes are given by the equation

$$\phi(\lambda, \mu, \nu) = 0, \tag{139}$$

or $$A\lambda^2 + B\mu^2 + C\nu^2 + 2D\mu\nu + 2E\nu\lambda + 2F\lambda\mu = 0;$$

whence, eliminating ν by means of the relation

$$a\lambda + b\mu + c\nu = 0,$$

we get, to determine the ratio $\lambda : \mu$,

$$A\lambda^2 + B\mu^2 + C\left(\frac{a\lambda + b\mu}{c}\right)^2 - 2(D\mu + E\lambda)\frac{a\lambda + b\mu}{c} + 2F\lambda\mu = 0,$$

which, on arranging the terms, becomes

$$(Ac^2 + Ca^2 - 2Eca)\lambda^2 + 2(Fc^2 - Ebc - Dca + Cab)\lambda\mu + (Cb^2 + Bc^2 - 2Dbc)\mu^2 = 0. \tag{140}$$

Now the roots of this equation are real and unequal, coincident, or imaginary, according as

$$(Fc^2 - Ebc - Dca + Cab)^2 \gtreqless (Ac^2 + Ca^2 - 2Eca)(Cb^2 + Bc^2 - 2Dbc).$$

This condition may be put into another form, for, multiplying out and dividing by c^2, we get, after arranging the terms,

$$(D^2 - BC)a^2 + (E^2 - CA)b^2 + (F^2 - AB)c^2$$

$$+ 2(AD - EF)bc + 2(BE - FD)ca + 2(CF - DE)ab \gtreqless 0,$$

that is (Prelim. chap. (D).)

$$-(A'a^2 + B'b^2 + C'c^2 + 2D'bc + 2E'ca + 2F'ab) \gtreqless 0;$$

or, according to the notation explained in Preliminary chap. (E),

$$-\phi(a, b, c)' \gtreqless 0 \tag{141}$$

or,
$$ {}^{a}\underset{c}{\Delta}{}^{b} \substack{>\\=\\<} 0. \tag{142}$$

Hence *the equation* $\phi(\alpha, \beta, \gamma) = 0$ *represents a Hyperbola, Parabola, or Ellipse, according as*

$\phi(a, b, c)'$ *is negative, zero,* or *positive,*

or as ${}^{a}\underset{c}{\Delta}{}^{b}$ *is positive, zero,* or *negative.*

126. *Equation of the chord joining the two points* $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ *on the conic* $\phi(\alpha, \beta, \gamma) = 0$.

The equation

$$\begin{aligned}&A(\alpha-\alpha_1)(\alpha-\alpha_2)+B(\beta-\beta_1)(\beta-\beta_2)+C(\gamma-\gamma_1)(\gamma-\gamma_2)\\ &+D\{(\beta-\beta_1)(\gamma-\gamma_2)+(\gamma-\gamma_1)(\beta-\beta_2)\}+E\{(\gamma-\gamma_1)(\alpha-\alpha_2)\\ &+(\alpha-\alpha_1)(\gamma-\gamma_2)\}+F\{(\alpha-\alpha_1)(\beta-\beta_2)+(\beta-\beta_1)(\alpha-\alpha_2)\}=A\alpha^2+B\beta_2\\ &+C\gamma^2+2D\beta\gamma+2E\gamma\alpha+2F\alpha\beta,\end{aligned} \tag{143}$$

is evidently satisfied by the co-ordinates of both the given points; and is linear, since the higher terms disappear on expansion.

It therefore represents the straight line on which the given points lie, and is the required equation of the chord of the conic.

To find the equation of the tangent to the conic $\phi(\alpha, \beta, \gamma) = 0$ *at the point* $(\alpha_1, \beta_1, \gamma_1)$.

127. *First method.*—Writing $\alpha_1, \beta_1, \gamma_1$ for $\alpha_2, \beta_2, \gamma_2$ respectively in (143) (since this is the same as making the two points on the curve, through which the chord passes, coincident), we get for the equation of the tangent at $(\alpha_1, \beta_1, \gamma_1)$

$$\begin{aligned}&A(\alpha-\alpha_1)^2+B(\beta-\beta_1)^2+C(\gamma-\gamma_1)^2+2D(\beta-\beta_1)(\gamma-\gamma_1)\\ &+2E(\gamma-\gamma_1)(\alpha-\alpha_1)+2F(\alpha-\alpha_1)(\beta-\beta_1)=A\alpha^2+B\beta^2+C\gamma^2\\ &+2D\beta\gamma+2E\gamma\alpha+2F\alpha\beta;\end{aligned}$$

or,

$$\begin{aligned}&A\alpha_1^2+B\beta_1^2+C\gamma_1^2+2D\beta_1\gamma_1+2E\gamma_1\alpha_1+2F\alpha_1\beta_1=2\{(A\alpha_1+F\beta_1+E\gamma_1)\alpha\\ &+(F\alpha_1+B\beta_1+D\gamma_1)\beta+(E\alpha_1+D\beta_1+C\gamma_1)\gamma\}.\end{aligned} \tag{144}$$

But the left-hand member of this equation vanishes, since $(\alpha_1, \beta_1, \gamma_1)$ is on the curve; we have therefore

$$(A\alpha_1 + F\beta_1 + E\gamma_1)\alpha + (F\alpha_1 + B\beta_1 + D\gamma_1)\beta + (E\alpha_1 + D\beta_1 + C\gamma_1)\gamma = 0,$$

which may be written in the abbreviated form

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha + \left(\frac{d\phi}{d\beta_1}\right)\beta + \left(\frac{d\phi}{d\gamma_1}\right)\gamma = 0, \tag{145}$$

or again (Prelim. chap. (*F*).), thus,

$$\left(\frac{d\phi}{d\alpha}\right)\alpha_1 + \left(\frac{d\phi}{d\beta}\right)\beta_1 + \left(\frac{d\varphi}{d\gamma}\right)\gamma_1 = 0; \tag{146}$$

the expressions which form the left-hand members of these two last equations being identically the same. Other methods of finding the equation of the tangent will now be given.

128. *Second method.*—Suppose

$$l\alpha + m\beta + n\gamma = 0 \tag{147}$$

to be the equation of the tangent.

Since it passes through the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_1 + d\alpha_1, \beta_1 + d\beta_1, \gamma_1 + d\gamma_1)$, we have the two equations

$$l\alpha_1 + m\beta_1 + n\gamma_1 = 0,$$
$$l d\alpha_1 + m d\beta_1 + n d\gamma_1 = 0,$$

which give, by cross multiplication,

$$\frac{l}{\begin{vmatrix} \beta_1, & \gamma_1 \\ d\beta_1, & d\gamma_1 \end{vmatrix}} = \frac{m}{\begin{vmatrix} \gamma_1, & \alpha_1 \\ d\gamma_1, & d\alpha_1 \end{vmatrix}} = \frac{n}{\begin{vmatrix} \alpha_1, & \beta_1 \\ d\alpha_1, & d\beta_1 \end{vmatrix}}. \tag{148}$$

Also, by Euler's Theorem of homogeneous functions,

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha_1 + \left(\frac{d\phi}{d\beta_1}\right)\beta_1 + \left(\frac{d\phi}{d\gamma_1}\right)\gamma_1 = 0,$$

and, from the equation of the conic,

$$\left(\frac{d\phi}{d\alpha_1}\right)d\alpha_1 + \left(\frac{d\phi}{d\beta_1}\right)d\beta_1 + \left(\frac{d\phi}{d\gamma_1}\right)d\gamma_1 = 0;$$

whence, as before,

$$\frac{\left(\frac{d\phi}{d\alpha_1}\right)}{\begin{vmatrix}\beta_1, & \gamma_1\\ d\beta_1, & d\gamma_1\end{vmatrix}}=\frac{\left(\frac{d\phi}{d\beta_1}\right)}{\begin{vmatrix}\gamma_1, & \alpha_1\\ d\gamma_1, & d\;{}_1\end{vmatrix}}=\frac{\left(\frac{d\phi}{d\gamma_1}\right)}{\begin{vmatrix}\alpha_1, & \beta_1\\ d\alpha_1, & d\beta_1\end{vmatrix}}. \tag{149}$$

From (148) and (149) it follows that

$$\frac{l}{\left(\frac{d\phi}{d\alpha_1}\right)}=\frac{m}{\left(\frac{d\phi}{a\beta_1}\right)}=\frac{n}{\left(\frac{d\phi}{d\gamma_1}\right)}; \tag{150}$$

and (147) becomes

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha+\left(\frac{d\phi}{d\beta_1}\right)\beta+\left(\frac{d\phi}{d\gamma_1}\right)\gamma=0,$$

which is the required equation.

129. *Third method.*—Taking the point $(\alpha_1, \beta_1, \gamma_1)$ as pole, we have (Art. 124) to determine r,

$$\phi(\lambda, \mu, \nu)r^2+\left\{\left(\frac{d\phi}{d\alpha_1}\right)\lambda+\left(\frac{d\phi}{d\beta_1}\right)\mu+\left(\frac{d\phi}{d\gamma_1}\right)\nu\right\}r+\phi(\alpha_1, \beta_1, \gamma_1)=0.$$

But, since the point $(\alpha_1, \beta_1, \gamma_1)$ is on the curve,

$$\phi(\alpha_1, \beta_1, \gamma_1)=0;$$

and, if the radius vector be a tangent, both values of r will $=0$ at the point; we must have, therefore,

$$\left(\frac{d\phi}{d\alpha_1}\right)\lambda+\left(\frac{d\phi}{d\beta_1}\right)\mu+\left(\frac{d\phi}{d\gamma_1}\right)\nu=0. \tag{151}$$

Hence, since (Art. 12. (7).) λ, μ, ν are proportional to $\alpha-\alpha_1$, $\beta-\beta_1$, $\gamma-\gamma_1$, respectively,

$$\left(\frac{d\phi}{d\alpha_1}\right)(\alpha-\alpha_1)+\left(\frac{d\phi}{d\beta_1}\right)(\beta-\beta_1)+\left(\frac{d\phi}{d\gamma_1}\right)(\gamma-\gamma_1)=0, \tag{152}$$

or, by Euler's Theorem of homogeneous functions,

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha + \left(\frac{d\phi}{d\beta_1}\right)\beta + \left(\frac{d\phi}{d\gamma_1}\right)\gamma = 0\,; \tag{153}$$

which is the equation of the tangent at $(\alpha_1, \beta_1, \gamma_1)$.

130. To determine the direction-cosines of the tangent, we have (Art. 129. (151).)

$$\left(\frac{d\phi}{d\alpha_1}\right)\lambda + \left(\frac{d\phi}{d\beta_1}\right)\mu + \left(\frac{d\phi}{d\gamma_1}\right)\nu = 0,$$

and the relation

$$a\lambda + b\mu + cn = 0:$$

whence

$$\frac{\lambda}{\begin{vmatrix} \left(\frac{d\phi}{d\beta_1}\right), & \left(\frac{d\phi}{d\gamma_1}\right) \\ b, & c \end{vmatrix}} = \frac{\mu}{\begin{vmatrix} \left(\frac{d\phi}{d\gamma_1}\right), & \left(\frac{d\phi}{d\alpha_1}\right) \\ c, & a \end{vmatrix}} = \frac{\nu}{\begin{vmatrix} \left(\frac{d\phi}{d\alpha_1}\right), & \left(\frac{d\phi}{d\beta_1}\right) \\ a, & b \end{vmatrix}},$$

and the symmetrical equations of the tangent may be formed at once.

131. *To find the condition that the straight line (l, m, n) should touch the conic $\phi(\alpha, \beta, \gamma) = 0$.*

Let $(\alpha_1, \beta_1, \gamma_1)$ be the point of contact. We must have (Art. 127. (145).)

$$\frac{\left(\frac{d\phi}{d\alpha_1}\right)}{l} = \frac{\left(\frac{d\phi}{d\beta_1}\right)}{m} = \frac{\left(\frac{d\phi}{d\gamma_1}\right)}{n} = -k \text{ (suppose).} \tag{154}$$

Therefore

$$\begin{aligned} A\alpha_1 + F\beta_1 + E\gamma_1 + lk &= 0, \\ F\alpha_1 + B\beta_1 + D\gamma_1 + mk &= 0, \\ E\alpha_1 + D\beta_1 + C\gamma_1 + nk &= 0. \end{aligned}$$

Also $$l\alpha_1 + m\beta_1 + n\gamma_1 = 0,$$

from the equation of the tangent. Whence, eliminating $\alpha_1, \beta_1, \gamma_1$ and k, we get for the required condition

$$\begin{vmatrix} A, & F, & E, & l \\ F, & B, & D, & m \\ E, & D, & C, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0; \qquad (155)$$

that is (Prelim. chap. (E),)

$$ {}^{l}\underset{n}{\Delta}{}^{m} = 0, \qquad (156)$$

or,

$$\phi(l, m, n)' = 0. \qquad (157)$$

Cor. Hence it appears (Art. 125) that *every parabola touches the straight line at infinity.*

132. *To find the equation of the normal to the conic* $\phi(\alpha, \beta, \gamma) = 0$ *at the point* $(\alpha_1, \beta_1, \gamma_1)$.

The equation of the tangent at the given point (Art. 127) is

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha + \left(\frac{d\phi}{d\beta_1}\right)\beta + \left(\frac{d\phi}{d\gamma_1}\right)\gamma = 0.$$

Hence the equations of the normal, which is a perpendicular to the tangent through the point, are (Art. 75. (88).)

$$\frac{\alpha - \alpha_1}{\left(\frac{d\phi}{d\alpha_1}\right) - \left(\frac{d\phi}{d\beta_1}\right)\cos C - \left(\frac{d\phi}{d\gamma_1}\right)\cos B} = \frac{\beta - \beta_1}{\left(\frac{d\phi}{d\beta_1}\right) - \left(\frac{d\phi}{d\gamma_1}\right)\cos A - \left(\frac{d\phi}{d\alpha_1}\right)\cos C}$$

$$= \frac{\gamma - \gamma_1}{\left(\frac{d\phi}{d\gamma_1}\right) - \left(\frac{d\phi}{d\alpha_1}\right)\cos B - \left(\frac{d\phi}{d\beta_1}\right)\cos A}. \qquad (158)$$

133. Hence also (Art. 76. (93).) *the equation of the normal at* $(\alpha_1, \beta_1, \gamma_1)$ *in the homogeneous form* is

$$\begin{vmatrix} \alpha, & \alpha_1, & \left(\frac{d\phi}{d\alpha_1}\right)-\left(\frac{d\phi}{d\beta_1}\right)\cos C-\left(\frac{d\phi}{d\gamma_1}\right)\cos B \\ \beta, & \beta_1, & \left(\frac{d\phi}{d\beta_1}\right)-\left(\frac{d\phi}{d\gamma_1}\right)\cos A-\left(\frac{d\phi}{d\alpha_1}\right)\cos C \\ \gamma, & \gamma_1, & \left(\frac{d\phi}{d\gamma_1}\right)-\left(\frac{d\phi}{d\alpha_1}\right)\cos B-\left(\frac{d\phi}{d\beta_1}\right)\cos A \end{vmatrix} = 0. \qquad (159)$$

134. *To find the equation of the polar of the point* $(\alpha_1, \beta_1, \gamma_1)$ *with respect to the conic* $\phi(\alpha, \beta, \gamma) = 0$.

Let $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ be the points of contact of tangents to the curve through $(\alpha_1, \beta_1, \gamma_1)$.

The equations of the tangents at the points $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ (Art. 127) are

$$\left(\frac{d\phi}{d\alpha}\right)\alpha_2+\left(\frac{d\phi}{d\beta}\right)\beta_2+\left(\frac{d\phi}{d\gamma}\right)\gamma_2=0,$$

$$\left(\frac{d\phi}{d\alpha}\right)\alpha_3+\left(\frac{d\phi}{d\beta}\right)\beta_3+\left(\frac{d\phi}{d\gamma}\right)\gamma_3=0;$$

and, since the point $(\alpha_1, \beta_1, \gamma_1)$ lies on both tangents, we have

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha_2+\left(\frac{d\phi}{d\beta_1}\right)\beta_2+\left(\frac{d\phi}{d\gamma_1}\right)\gamma_2=0$$

and
$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha_3+\left(\frac{d\phi}{d\beta_1}\right)\beta_3+\left(\frac{d\phi}{d\gamma_1}\right)\gamma_3=0;$$

which shew that

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha+\left(\frac{d\phi}{d\beta_1}\right)\beta+\left(\frac{d\phi}{d\gamma_1}\right)\gamma=0 \qquad (160)$$

is the equation of the line on which the two points of contact lie, and therefore represents the polar of the point $(\alpha_1, \beta_1, \gamma_1)$.

135. If the polar be defined as the locus of points whose distances from $(\alpha_1, \beta_1, \gamma_1)$ are harmonic means between the radii vectores drawn from that point, in the same direction, to the curve, the following method may be employed.

The equation of the conic referred to the given point as pole is

$$\phi(\lambda, \mu, \nu) r^2 + \left\{ \left(\frac{d\phi}{d\alpha_1}\right)\lambda + \left(\frac{d\phi}{d\beta_1}\right)\mu + \left(\frac{d\phi}{d\gamma_1}\right)\nu \right\} r + \phi(\alpha_1, \beta_1, \gamma_1) = 0;$$

and if R be the Harmonic mean between the roots of this equation, we shall have

$$R\left\{ \left(\frac{d\phi}{d\alpha_1}\right)\lambda + \left(\frac{d\phi}{d\beta_1}\right)\mu + \left(\frac{d\phi}{d\gamma_1}\right)\nu \right\} + 2\phi(\alpha_1, \beta_1, \gamma_1) = 0;$$

or, since $R\lambda = \alpha - \alpha_1$, $R\mu = \beta - \beta_1$, $R\nu = \gamma - \gamma_1$ (Art. 12. (7).),

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha + \left(\frac{d\phi}{d\beta_1}\right)\beta + \left(\frac{d\phi}{d\gamma_1}\right)\gamma = \left[\left(\frac{d\phi}{d\alpha_1}\right)\alpha_1 + \left(\frac{d\phi}{d\beta_1}\right)\beta_1 + \left(\frac{d\phi}{d\gamma_1}\right)\gamma_1\right] - 2\phi(\alpha_1, \beta_1, \gamma_1)$$

$$= 0, \text{ by Euler's Theorem.}$$

136. *To find the pole of the straight line* (l_1, m_1, n_1) *with respect to the conic* $\phi(\alpha, \beta, \gamma) = 0$.

If $(\alpha_1, \beta_1, \gamma_1)$ be the co-ordinates of the pole we must have, by equation (160)

$$\frac{\left(\frac{d\phi}{d\alpha_1}\right)}{l_1} = \frac{\left(\frac{d\phi}{d\beta_1}\right)}{m_1} = \frac{\left(\frac{d\phi}{d\gamma_1}\right)}{n_1} = -k \text{ (suppose)}: \qquad (161)$$

that is,

$$\begin{aligned} A\alpha_1 + F\beta_1 + E\gamma_1 + l_1 k &= 0, \\ F\alpha_1 + B\beta_1 + D\gamma_1 + m_1 k &= 0, \qquad (162) \\ E\alpha_1 + D\beta_1 + C\gamma_1 + n_1 k &= 0; \end{aligned}$$

these, with the equation

$$a\alpha_1 + b\beta_1 + c\gamma_1 = 2S,$$

completely determine the values of $(\alpha_1, \beta_1, \gamma_1)$. (See Art. 137.)

137. *To find the co-ordinates of the centre of the conic* $\phi(\alpha, \beta, \gamma) = 0$.

Let $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ be the co-ordinates of the centre; the polar equation of the conic, referred to this point, will (Art. 13) be

$$\phi(\lambda, \mu, \nu)r^2 + \left\{ \left(\frac{d\phi}{d\bar{\alpha}}\right)\lambda + \left(\frac{d\phi}{d\bar{\beta}}\right)\mu + \left(\frac{d\phi}{d\bar{\gamma}}\right)\nu \right\} r + \phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = 0. \tag{163}$$

Hence, as the two values of r must be, for *any* direction, equal in magnitude but of opposite sign, we must have

$$\left(\frac{d\phi}{d\bar{\alpha}}\right)\lambda + \left(\frac{d\phi}{d\bar{\beta}}\right)\mu + \left(\frac{d\phi}{d\bar{\gamma}}\right)\nu = 0, \tag{164}$$

where λ, μ, ν are not independent but connected by the relation

$$a\lambda + b\mu + c\nu = 0. \tag{165}$$

Now (164) and (165) are true simultaneously for *all* values of λ, μ, ν: they give therefore

$$\frac{\left(\frac{d\phi}{d\bar{\alpha}}\right)}{a} = \frac{\left(\frac{d\phi}{d\bar{\beta}}\right)}{b} = \frac{\left(\frac{d\phi}{d\bar{\gamma}}\right)}{c} = \frac{\left(\frac{d\phi}{d\bar{\alpha}}\right)\bar{\alpha} + \left(\frac{d\phi}{d\bar{\beta}}\right)\bar{\beta} + \left(\frac{d\phi}{d\bar{\gamma}}\right)\bar{\gamma}}{a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma}}$$

$$= \frac{\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})}{S}, \tag{166}$$

(Prelim. chap. (A), and Euler's Theorem).

Equations (166) may be written

$$\left.\begin{aligned} A\bar{\alpha} + F\bar{\beta} + E\bar{\gamma} - a\frac{\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})}{2S} &= 0; \\ F\bar{\alpha} + B\bar{\beta} + D\bar{\gamma} - b\frac{\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})}{2S} &= 0, \\ E\bar{\alpha} + D\bar{\beta} + C\bar{\gamma} - c\frac{\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})}{2S} &= 0; \end{aligned}\right\} \tag{167}$$

whence we get (Prelim. chap. (E).)

$$\frac{\bar{\alpha}}{\begin{vmatrix} F, & E, & a \\ B, & D, & b \\ D, & C, & c \end{vmatrix}} = \frac{\bar{\beta}}{-\begin{vmatrix} A, & E, & a \\ F, & D, & b \\ E, & C, & c \end{vmatrix}} = \frac{\bar{\gamma}}{\begin{vmatrix} A, & F, & a \\ F, & B, & b \\ E, & D, & c \end{vmatrix}} = \frac{-\frac{\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})}{2S}}{-\begin{vmatrix} A, & F, & E \\ F, & B, & D \\ E, & D, & C \end{vmatrix}}. \tag{168}$$

138. Equations (168) of the last Article may be put into the form

$$\frac{\bar{a}}{\begin{vmatrix} a, & F, & F \\ b, & B, & D \\ c, & D, & C \end{vmatrix}} = \frac{\bar{\beta}}{\begin{vmatrix} A, & a, & E \\ F, & b, & D \\ E, & c, & C \end{vmatrix}} = \frac{\bar{\gamma}}{\begin{vmatrix} A, & F, & a \\ F, & B, & b \\ E, & D & c \end{vmatrix}} = \frac{\frac{\phi(\bar{a}, \bar{\beta}, \bar{\gamma})}{2S}}{\begin{vmatrix} A, & F, & E \\ F, & B, & D \\ E, & D, & C \end{vmatrix}}; \qquad (169)$$

or, if we expand the determinants in terms of their first minors and write Δ for the denominator of the last member of (169),

$$\frac{\bar{a}}{aA'+bF'+cE'} = \frac{\bar{\beta}}{aF'+bB'+cD'} = \frac{\bar{\gamma}}{aE'+bD'+cC'} = \frac{\phi(\bar{a}, \bar{\beta}, \bar{\gamma})}{\Delta \,.\, S}$$

(Prelim. chap. (*A*).); (170)

$$\text{also} = \frac{a\bar{a}+b\bar{\beta}+c\bar{\gamma}}{A'a^2+B'b^2+C'c^2+2D'bc+2E'ca+2F'ab} = \frac{2S}{\phi(a, b, c)'}$$

$$= \frac{2S}{-{}^{a}\underset{c}{\Delta}{}^{b}} \quad \text{(Prelim. chap. } (E)\text{.)}, \qquad (171)$$

whence

$$\phi(\bar{a}, \bar{\beta}, \bar{\gamma}) = -4S^2 \frac{\Delta}{{}^{a}\underset{c}{\Delta}{}^{b}}; \qquad (172)$$

a relation which will be useful hereafter.

139. A comparison of (166) with the equations (161) of Art. 136 shews that *in every conic the centre is the pole of the straight line at infinity* (Art. 52). And in finding the co-ordinates of the centre of the conic $\phi(a, \beta, \gamma) = 0$, the student may, if he please, start with this property of the centre, and, deducing the equations (167) as in Art. 136, proceed as in the last Article.

Again, since (Art. 125. (142).) ${}^{a}\underset{c}{\Delta}{}^{b} = 0$ when the conic is a parabola, it appears from (171) that *the centre of a parabola lies on the straight line at infinity.*

140. *To find the equation of the pair of tangents which may be drawn to the conic* $\phi(a, \beta, \gamma) = 0$, *through the point* (a_0, β_0, γ_0).

The equation

$$\phi(\lambda, \mu, \nu)r^2 + \left\{ \left(\frac{d\phi}{da_0}\right)\lambda + \left(\frac{d\phi}{d\beta_0}\right)\mu + \left(\frac{d\phi}{d\gamma_0}\right)\nu \right\} r + \phi(a_0, \beta_0, \gamma_0) = 0$$

gives the length of the radius vector drawn to the curve in any direction from the given point. It will have equal roots, and the radius vector, whose equation is

$$\frac{a - a_0}{\lambda} = \frac{\beta - \beta_0}{\mu} = \frac{\gamma - \gamma_0}{\nu} = r, \tag{173}$$

will touch the curve, provided that

$$\left[\left(\frac{d\phi}{da_0}\right)\lambda + \left(\frac{d\phi}{d\beta_0}\right)\mu + \left(\frac{d\phi}{d\gamma_0}\right)\nu\right]^2 = 4\phi(a_0, \beta_0, \gamma_0)\,\phi(\lambda, \mu, \nu). \tag{174}$$

Hence, substituting for λ, μ, ν from (173), we get for the equation of the pair of tangents through (a_0, β_0, γ_0)

$$\begin{aligned}\left[\left(\frac{d\phi}{da_0}\right)(a-a_0) + \left(\frac{d\phi}{d\beta_0}\right)(\beta-\beta_0) + \left(\frac{d\phi}{d\gamma_0}\right)(\gamma-\gamma_0)\right]^2 \\ = 4\phi(a_0, \beta_0, \gamma_0)\,\phi(a-a_0, \beta-\beta_0, \gamma-\gamma_0) \qquad (175) \\ = 4\phi(a_0, \beta_0, \gamma_0)\left[\phi(a, \beta, \gamma) - \left\{\left(\frac{d\phi}{da}\right)a_0\right.\right. \\ \left.\left. + \left(\frac{d\phi}{d\beta}\right)\beta_0 + \left(\frac{d\phi}{d\gamma}\right)\gamma_0\right\} + \phi(a_0, \beta_0, \gamma_0)\right];\end{aligned}$$

which becomes, since, by Euler's Theorem,

$$\left(\frac{d\phi}{da_0}\right)a_0 + \left(\frac{d\phi}{d\beta_0}\right)\beta_0 + \left(\frac{d\phi}{d\gamma_0}\right)\gamma_0 = 2\phi(a_0, \beta_0, \gamma_0),$$

$$\left[\left(\frac{d\phi}{da_0}\right)a + \left(\frac{d\phi}{d\beta_0}\right)\beta + \left(\frac{d\phi}{d\gamma_0}\right)\gamma\right]^2 = 4\phi(a_0, \beta_0, \gamma_0)\,\phi(a, \beta, \gamma). \tag{176}$$

The form of (176) shews (Art. 170. (*C*).) that its locus is a curve of the second order having double contact with the conic $\phi(a, \beta, \gamma) = 0$ at the points where it is met by the polar of (a_0, β_0, γ_0).

141. Equation (176) of the last Article may be written in the form

$$(C'\beta_0^2+B'\gamma_0^2-2D'\beta_0\gamma_0)\alpha^2+(A'\gamma_0^2+C'\alpha_0^2-2E'\gamma_0\alpha_0)\beta^2+(B'\alpha_0^2+A'\beta_0^2-2F'\alpha_0\beta_0)\gamma^2$$
$$-2(D'\alpha_0^2+A'\beta_0\gamma_0-F'\gamma_0\alpha_0-E'\alpha_0\beta_0)\beta\gamma-2(E'\beta_0^2+B'\gamma_0\alpha_0-D'\alpha_0\beta_0-F'\beta_0\gamma_0)\gamma\alpha$$
$$-2(F'\gamma_0^2+C'\alpha_0\beta_0-E'\beta_0\gamma_0-D'\gamma_0\alpha_0)\alpha\beta=0. \qquad (177)$$

142. *To find the locus of the point of intersection of tangents to the conic $\phi(\alpha, \beta, \gamma) = 0$ which cut each other at right angles.*

Equation (174) of Art. 140 gives a relation between the direction-cosines of the pair of tangents which can be drawn to the curve through the point $(\alpha_0, \beta_0, \gamma_0)$: but these, if the tangents intersect at right angles, must (Art. 33) be of the form $(\cos a_1, \cos b_1, \cos c_1)$, $(\sin a_1, \sin b_1, \sin c_1)$; hence we shall have

$$\left[\left(\frac{d\phi}{d\alpha_0}\right)\cos a_1+\left(\frac{d\phi}{d\beta_0}\right)\cos b_1+\left(\frac{d\phi}{d\gamma_0}\right)\cos c_1\right]^2 = 4\,\phi(\alpha_0, \beta_0, \gamma_0)\,[A\cos^2 a_1$$
$$+ B\cos^2 b_1 + C\cos^2 c_1 + 2D\cos b_1\cos c_1 + \&c. . .],$$

and

$$\left[\left(\frac{d\phi}{d\alpha_0}\right)\sin a_1+\left(\frac{d\phi}{d\beta_0}\right)\sin b_1+\left(\frac{d\phi}{d\gamma_0}\right)\sin c_1\right]^2 = 4\,\phi(\alpha_0, \beta_0, \gamma_0)\,[A\sin^2 a_1$$
$$+ B\sin^2 b_1 + C\sin^2 c_1 + 2D\sin b_1\sin c_1 + \&c. . .];$$

adding (Art. 30. (22).), and omitting the suffixes, we get for the equation of the locus

$$\left(\frac{d\phi}{d\alpha}\right)^2+\left(\frac{d\phi}{d\beta}\right)^2+\left(\frac{d\phi}{d\gamma}\right)^2-2\left(\frac{d\phi}{d\beta}\right)\left(\frac{d\phi}{d\gamma}\right)\cos A-2\left(\frac{d\phi}{d\gamma}\right)\left(\frac{d\phi}{d\alpha}\right)\cos B$$
$$-2\left(\frac{d\phi}{d\alpha}\right)\left(\frac{d\phi}{d\beta}\right)\cos C = 4\,\phi(\alpha, \beta, \gamma)\,[A+B+C-2\,D\cos A$$
$$-2\,E\cos B-2\,F\cos C]. \qquad (178)$$

143. The equation of the last Article may also be written in the form

$$(B' + C' + 2D' \cos A)\alpha^2 + (C' + A' + 2E' \cos B)\beta^2 + (A' + B' + 2F' \cos C)\gamma^2 - 2(D' - A' \cos A + F' \cos B + E' \cos C)\beta\gamma - 2(E' - B' \cos B + D' \cos C + F' \cos A)\gamma\alpha - 2(F' - C' \cos C + E' \cos A + D' \cos B)\alpha\beta = 0. \qquad (179)$$

This, therefore, is the equation of the *director* of the conic $\phi(\alpha, \beta, \gamma) = 0$, and will be found to satisfy the condition for a circle. (See Art. 149.)

144. *To find the equation of the asymptotes of the conic* $\phi(\alpha, \beta, \gamma) = 0$.

The asymptotes being a pair of tangents which have the line at infinity for their chord of contact, their equation must be of the form

$$\phi(\alpha, \beta, \gamma) = k(a\alpha + b\beta + c\gamma)^2$$
$$= 4kS^2.$$

Since they pass through the centre of the curve, we have,

$$\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = 4kS^2;$$

and consequently their equation is

$$\phi(\alpha, \beta, \gamma) = \phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma}). \qquad (180)$$

This equation may be written in the homogeneous form

$$4S^2 \phi(\alpha, \beta, \gamma) = \phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})(a\alpha + b\beta + c\gamma)^2; \qquad (181)$$

or, again (Art. 138. (172).),

$${}^{a}\underset{c}{\Delta}{}^{b} \phi(\alpha, \beta, \gamma) + \Delta (a\alpha + b\beta + c\gamma)^2 = 0. \qquad (182)$$

145. Equation (180) of the last Article may of course be obtained directly from the polar equation of the conic. For, as in Art. 125, we have the directions of the asymptotes given by the equation

$$\phi(\lambda, \mu, \nu) = 0;$$

hence their equation is

$$\phi(\alpha-\bar{\alpha},\ \beta-\bar{\beta},\ \gamma-\bar{\gamma})=0,$$

or

$$\phi(\alpha,\beta,\gamma)-\left\{\left(\frac{d\phi}{d\bar{\alpha}}\right)\alpha+\left(\frac{d\phi}{d\bar{\beta}}\right)\beta+\left(\frac{d\phi}{d\bar{\gamma}}\right)\gamma\right\}+\phi(\bar{\alpha},\bar{\beta},\bar{\gamma})=0. \tag{183}$$

But, since $(\bar{\alpha},\bar{\beta},\bar{\gamma})$ is the centre, we have (Art. 137. (164).)

$$\left(\frac{d\phi}{d\bar{\alpha}}\right)\lambda+\left(\frac{d\phi}{d\bar{\beta}}\right)\mu+\left(\frac{d\phi}{d\bar{\gamma}}\right)\nu=0,$$

or $$\left(\frac{d\phi}{d\bar{\alpha}}\right)(\alpha-\alpha)+\left(\frac{d\phi}{d\bar{\beta}}\right)(\beta-\bar{\beta})+\left(\frac{d\phi}{d\bar{\gamma}}\right)(\gamma-\bar{\gamma})=0;$$

that is,

$$\left(\frac{d\phi}{d\bar{\alpha}}\right)\alpha+\left(\frac{d\phi}{d\bar{\beta}}\right)\beta+\left(\frac{d\phi}{d\bar{\gamma}}\right)\gamma=\left(\frac{d\phi}{d\bar{\alpha}}\right)\bar{\alpha}+\left(\frac{d\phi}{d\bar{\beta}}\right)\bar{\beta}+\left(\frac{d\phi}{d\bar{\gamma}}\right)\bar{\gamma}$$

$$=2\,\phi(\bar{\alpha},\bar{\beta},\bar{\gamma}),\text{ by Euler's Theorem ;}$$

and (183) becomes,

$$\phi(\alpha,\beta,\gamma)=\phi(\bar{\alpha},\bar{\beta},\bar{\gamma}).$$

146. *To find the condition that the equation $\phi(\alpha,\beta,\gamma)=0$ may represent a pair of right lines.*

Let $(\bar{\alpha},\bar{\beta},\bar{\gamma})$ be the point of intersection of the two lines. This point being the centre of the conic, and therefore the pole of the line at infinity, we have, as in Art. 136. (161),

$$\frac{\left(\frac{d\phi}{d\bar{\alpha}}\right)}{a}=\frac{\left(\frac{d\phi}{d\bar{\beta}}\right)}{b}=\frac{\left(\frac{d\phi}{d\bar{\gamma}}\right)}{c}=\frac{\phi(\bar{\alpha},\bar{\beta},\bar{\gamma})}{S}. \tag{184}$$

But, in this case, the centre lies *on* the conic; so that

$$\phi(\bar{\alpha},\bar{\beta},\bar{\gamma})=0,$$

and we must have simultaneously

$$\left(\frac{d\phi}{d\bar{\alpha}}\right) = 0, \quad \left(\frac{d\phi}{d\bar{\beta}}\right) = 0, \quad \left(\frac{d\phi}{d\bar{\gamma}}\right) = 0; \tag{185}$$

that is to say,

$$\begin{aligned} A\bar{\alpha} + F\bar{\beta} + E\bar{\gamma} &= 0, \\ F\bar{\alpha} + B\bar{\beta} + D\bar{\gamma} &= 0, \\ E\bar{\alpha} + D\bar{\beta} + C\bar{\gamma} &= 0; \end{aligned} \tag{186}$$

and the condition that these should co-exist is

$$\begin{vmatrix} A, & F, & E \\ F, & B, & D \\ E, & D, & C \end{vmatrix} = 0, \tag{187}$$

or, (Prelim. chap. (*D*),)

$$\Delta = 0. \tag{188}$$

147. The condition of the last Article may be otherwise obtained, as follows. If $\phi(\alpha, \beta, \gamma) = 0$ represents a pair of right lines, $\phi(\alpha, \beta, \gamma)$ must be the product of two linear factors u and v (suppose): so that

$$\phi(\alpha, \beta, \gamma) = uv = 0,$$

and we shall have

$$\left(\frac{d\phi}{d\alpha}\right) = u\left(\frac{dv}{d\alpha}\right) + v\left(\frac{du}{d\alpha}\right),$$

$$\left(\frac{d\phi}{d\beta}\right) = u\left(\frac{dv}{d\beta}\right) + v\left(\frac{du}{d\beta}\right),$$

$$\left(\frac{d\phi}{d\gamma}\right) = u\left(\frac{dv}{d\gamma}\right) + v\left(\frac{du}{d\gamma}\right);$$

whence it appears that any values of α, β, γ which satisfy $u = 0$, $v = 0$ simultaneously, will also make

$$\left(\frac{d\phi}{d\alpha}\right) = \left(\frac{d\phi}{d\beta}\right) = \left(\frac{d\phi}{d\gamma}\right) = 0,$$

and we obtain, as before,

$$\Delta = 0.$$

148. *To find the condition that the equation $\phi(\alpha, \beta, \gamma) = 0$ should represent a parabola.*

If the conic be a parabola, the straight line at infinity

$$a\alpha + b\beta + c\gamma = 0 \tag{189}$$

is a tangent to it (Art. 131. Cor.). Let $(\alpha_1, \beta_1, \gamma_1)$ be the point of contact. We shall have (Art. 131. (154).)

$$\frac{\left(\frac{d\phi}{d\alpha_1}\right)}{a} = \frac{\left(\frac{d\phi}{d\beta_1}\right)}{b} = \frac{\left(\frac{d\phi}{d\gamma_1}\right)}{c} = -k \text{ (say)},$$

whence
$$A\alpha_1 + F\beta_1 + E\gamma_1 + ak = 0,$$
$$F\alpha_1 + B\beta_1 + D\gamma_1 + bk = 0,$$
$$E\alpha_1 + D\beta_1 + C\gamma_1 + ck = 0,$$

also by (189)
$$a\alpha_1 + b\beta_1 + c\gamma_1 = 0:$$

and eliminating $\alpha_1, \beta_1, \gamma_1$ between these equations we have for the required condition

$${}^{a}\underset{c}{\Delta}{}^{b} = 0, \tag{190}$$

which (Prelim. chap. (E).) may also be written in the form

$$\phi(a, b, c)' = 0. \tag{191}$$

149. *To find the conditions that the equation $\phi(\alpha, \beta\ \gamma) = 0$ may represent a circle.*

The polar equation of the conic, referred to the centre, since the co-efficient of r vanishes, gives

$$\begin{aligned}\phi(\lambda, \mu, \nu)r^2 &= -\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \\ &= \text{a constant.}\end{aligned} \tag{192}$$

Hence, if $\rho_\alpha, \rho_\beta, \rho_\gamma$ be the lengths of the semi-diameters respectively parallel to the three sides of the triangle of reference, we shall have

$$\phi(0, -\sin C, \sin B)\rho_\alpha^2 = \phi(\sin C, 0, -\sin A)\rho_\beta^2 = \phi(-\sin B, \sin A, 0)\rho_\gamma^2. \tag{193}$$

But, in the case of the circle,

$$\rho_\alpha = \rho_\beta = \rho_\gamma :$$

hence we have for the required conditions,

$$\phi(0, -\sin C, \sin B) = \phi(\sin C, 0, -\sin A) = \phi(-\sin B, \sin A, 0). \tag{194}$$

Or, if we write them at full length,

$$\begin{aligned} B\sin^2 C + C\sin^2 B - 2D\sin B\sin C &= C\sin^2 A + A\sin^2 C - 2E\sin C\sin A \\ &= A\sin^2 B + B\sin^2 A - 2F\sin A\sin B. \end{aligned} \tag{195}$$

150. *To find the condition that the equation* $\phi(\alpha, \beta, \gamma) = 0$ *should represent a rectangular hyperbola.*

The directions of the asymptotes (Art. 125) are given by the equation

$$\phi(\lambda, \mu, \nu) = 0.$$

But if these are mutually perpendicular, the two sets of values for λ, μ, ν, given by this equation, will be (Art. 33) of the form $(\cos a_1, \cos b_1, \cos c_1)$ and $(\sin a_1, \sin b_1, \sin c_1)$.

Hence
$$\phi(\cos a_1, \cos b_1, \cos c_1) = 0,$$
$$\phi(\sin a_1, \sin b_1, \sin c_1) = 0;$$

adding and remembering the relations (22) of Art. 30, we get, as in Art. 72,

$$A + B + C - 2D\cos A - 2E\cos B - 2F\cos C = 0 \tag{196}$$

for the required condition.

151. *To find the conditions that two conics whose equations are* $\phi(\alpha, \beta, \gamma) = 0$, $f(\alpha, \beta, \gamma) = 0$, *should be similar and similarly situated.*

Let $\rho_\alpha, \rho_\beta, \rho_\gamma$ and $r_\alpha, r_\beta, r_\gamma$ be the central radii vectores of the two conics, drawn parallel to the sides of the triangle of reference: we have, as in (193) Art. 149,

$$\phi(0, -\sin C, \sin B)\rho_\alpha^2 = \phi(\sin C, 0, -\sin A)\rho_\beta^2 = \phi(-\sin B, \sin A, 0)\rho_\gamma^2;$$

$$f(0, -\sin C, \sin B)r_\alpha^2 = f(\sin C, 0, -\sin A)r_\beta^2 = f(-\sin B, \sin A, 0)r_\gamma^2.$$

But, if the two conics are similar and similarly situated, we must have

$$\frac{\rho_\alpha}{r_\alpha} = \frac{\rho_\beta}{r_\beta} = \frac{\rho_\gamma}{r_\gamma}:$$

hence the required condition is

$$\frac{\phi(0, -\sin C, \sin B)}{f(0, -\sin C, \sin B)} = \frac{\phi(\sin C, 0, -\sin A)}{f(\sin C, 0, -\sin A)} = \frac{\phi(-\sin B, \sin A, 0)}{f(-\sin B, \sin A, 0)}; \tag{197}$$

in other words, the quantities

$$B\sin^2 C + C\sin^2 B - 2D\sin B\sin C,$$

$$C\sin^2 A + A\sin^2 C - 2E\sin C\sin A,$$

and $$A\sin^2 B + B\sin^2 A - 2F\sin A\sin B,$$

must be to each other in a constant ratio.

152. *To find the direction of the axis of the parabola whose equation is* $\phi(\alpha, \beta, \gamma) = 0$.

The equation

$$(Ac^2 + Ca^2 - 2Eca)\lambda^2 + 2(Fc^2 - Ebc - Dca - Cab)\lambda\mu + (Cb^2 + Bc^2 - 2Dbc)\mu^2 = 0,$$

(Art. 125. (140).) gives the value of the ratio λ : μ for the directions of the asymptotes.

If $\phi(\alpha, \beta, \gamma) = 0$ be a parabola, the asymptotes will be co-incident and in the direction of the principal diameter, and the above equation will give

$$\frac{\lambda}{\sqrt{\{Cb^2+Bc^2-2Dbc\}}} = \frac{\mu}{\sqrt{\{Ac^2+Ca^2-2Eca\}}} = \frac{\nu}{\sqrt{\{Ba^2+Ab^2-2Fab\}}}; \tag{198}$$

the last member following by symmetry.

These equations, therefore, determine the direction of the axis of the parabola.

153. *To find the equation of the circle, of radius ρ, whose centre is at the point $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$.*

Let (α, β, γ) be any point on the curve; then, since its distance from the centre $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is constant and $= \rho$, we have (Art. 45. (48).)

$$a(\beta-\beta)(\gamma-\bar{\gamma})+b(\gamma-\bar{\gamma})(\alpha-\bar{\alpha})+c(\alpha-\bar{\alpha})(\beta-\bar{\beta}) + \frac{4S^2\rho^2}{abc} = 0; \tag{199}$$

which is the required equation.

154. *To find the radius of the circle whose equation is $\phi(\alpha, \beta, \gamma) = 0$.*

The length of the central radius vector of a conic, in any direction (λ, μ, ν) is given (192), (172), by the equation

$$\begin{aligned} \phi(\lambda, \mu, \nu)r^2 &= -\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \\ &= 4S^2 \frac{\Delta}{\overset{a\ b}{\underset{c}{\Delta}}}. \end{aligned} \tag{200}$$

If therefore r_1, r_2 be the lengths of any two radii $(\cos a_1, \cos b_1, \cos c_1)$, $(\sin a_1, \sin b_1, \sin c_1)$, which are at right angles to each other, we shall have

$$\phi(\cos a_1, \cos b_1, \cos c_1) = \frac{4S^2\Delta}{\overset{a\ b}{\underset{c}{\Delta}}} \cdot \frac{1}{r_1^2},$$

and

$$\phi(\sin a_1, \sin b_1, \sin c_1) = \frac{4S^2\Delta}{\overset{a\ b}{\underset{c}{\Delta}}} \cdot \frac{1}{r_2^2};$$

whence, adding, we get

$$A+B+C-2D\cos A-2E\cos B-2F\cos C=\frac{4S^2\Delta}{{}^{a}\underset{c}{\Delta}{}^{b}}\left[\frac{1}{r_1^2}+\frac{1}{r_2^2}\right], \tag{201}$$

a relation which holds for any pair of rectangular central radii of a conic.

But, in the case of a circle,

$$r_1=r_2=\rho \text{ (suppose).}$$

Hence the radius of the circle $\phi(\alpha, \beta, \gamma)=0$ is given by the equation

$$\frac{1}{\rho^2}=\frac{{}^{a}\underset{c}{\Delta}{}^{b}}{8S^2\Delta}\left[A+B+C-2D\cos A-2E\cos B-2F\cos C\right]. \tag{202}$$

155. From equation (201) the condition for a rectangular hyperbola (Art. 150. (196).) may be easily deduced. Making $r_2^2=-r_1^2$ (since one pair of the principal semi-axes is imaginary), we get

$$A+B+C-2D\cos A-2E\cos B-2F\cos C=0. \tag{203}$$

156. *To find the area of the circle whose equation is* $\phi(\alpha, \beta, \gamma)=0$.

Using the value of the radius given in (202), we have;—

$$\text{Area of } \odot=\pi\rho^2$$

$$=\frac{8\pi S^2\Delta}{{}^{a}\underset{c}{\Delta}{}^{b}}\left[A+B+C-2D\cos A-2E\cos B-2F\cos C\right]^{-1}. \tag{204}$$

157. *To find the equation of the diameter which bisects all chords whose direction cosines are* λ, μ, ν.

If $(\alpha_0, \beta_0, \gamma_0)$ be *any* point, its distance from the curve in any direction is determined by the quadratic

$$\phi(\lambda, \mu, \nu)\, r^2+\left\{\left(\frac{d\phi}{d\lambda}\right)\alpha_0+\left(\frac{d\phi}{d\mu}\right)\beta_0+\left(\frac{d\phi}{d\nu}\right)\gamma_0\right\} r+\phi(\alpha_0, \beta_0, \gamma_0)=0.$$

If therefore $(\alpha_0, \beta_0, \gamma_0)$ be the middle point of a chord of the given system, we must have

$$\left(\frac{d\phi}{d\lambda}\right)\alpha_0 + \left(\frac{d\phi}{d\mu}\right)\beta_0 + \left(\frac{d\phi}{d\nu}\right)\gamma_0 = 0.$$

Hence $(\alpha_0, \beta_0, \gamma_0)$ lies on the right line whose equation is

$$\left(\frac{d\phi}{d\lambda}\right)\alpha + \left(\frac{d\phi}{d\mu}\right)\beta + \left(\frac{d\phi}{d\nu}\right)\gamma = 0: \tag{205}$$

this therefore is the equation of the diameter or locus of middle points.

Cor. I. Hence

$$\left(\frac{d\phi}{d\lambda}\right)\alpha + \left(\frac{d\phi}{d\mu}\right)\beta + \left(\frac{d\phi}{d\nu}\right)\gamma = 0 \tag{206}$$

is the equation of the diameter which is conjugate to

$$\frac{\alpha - \bar{\alpha}}{\lambda} = \frac{\beta - \bar{\beta}}{\mu} = \frac{\gamma - \bar{\gamma}}{\nu} = r. \tag{207}$$

Cor. II. Its direction-cosines being formed as in Art. 40, we may write the equation of the diameter which is conjugate to

$$\frac{\alpha - \bar{\alpha}}{\lambda} = \frac{\beta - \bar{\beta}}{\mu} = \frac{\gamma - \bar{\gamma}}{\nu} = r, \tag{208}$$

in the symmetrical form

$$\frac{\alpha - \bar{\alpha}}{\begin{vmatrix} b, & c \\ \left(\frac{d\phi}{d\mu}\right), & \left(\frac{d\phi}{d\nu}\right) \end{vmatrix}} = \frac{\beta - \bar{\beta}}{\begin{vmatrix} c, & a \\ \left(\frac{d\phi}{d\nu}\right), & \left(\frac{d\phi}{d\lambda}\right) \end{vmatrix}} = \frac{\gamma - \bar{\gamma}}{\begin{vmatrix} a, & b \\ \left(\frac{d\phi}{d\lambda}\right), & \left(\frac{d\phi}{d\mu}\right) \end{vmatrix}}. \tag{209}$$

Cor. III. Again, it follows from Art. 157, that two straight lines, whose equations are of the form

$$l\alpha + m\beta + n\gamma = 0 \tag{210}$$

and $$\begin{vmatrix} b, & c \\ m, & n \end{vmatrix}\left(\frac{d\phi}{da}\right)+\begin{vmatrix} c, & a \\ n, & l \end{vmatrix}\left(\frac{d\phi}{d\beta}\right)+\begin{vmatrix} a, & b \\ l, & m \end{vmatrix}\left(\frac{d\phi}{d\gamma}\right)=0, \qquad (211)$$

are parallel to conjugate diameters.

158. *To find the condition that the straight lines* $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$ *may be parallel to conjugate diameters of the conic* $\phi(a, \beta, \gamma) = 0$.

It was shewn in the last Article that the straight line

$$\left(\frac{d\phi}{d\lambda_1}\right)a+\left(\frac{d\phi}{d\mu_1}\right)\beta+\left(\frac{d\phi}{d\nu_1}\right)\gamma=0$$

is conjugate to $(\lambda_1, \mu_1, \nu_1)$. Hence we have

$$\frac{l_2}{\left(\frac{d\phi}{d\lambda_1}\right)}=\frac{m_2}{\left(\frac{d\phi}{d\mu_1}\right)}=\frac{n_2}{\left(\frac{d\phi}{d\nu_1}\right)}, \quad \text{(Art. 29,)}$$

and the relation

$$l_2\lambda_2+m_2\mu_2+n_2\nu_2=0 \quad \text{(Art. 31. (26).)}$$

becomes

$$\left(\frac{d\phi}{d\lambda_1}\right)\lambda_2+\left(\frac{d\phi}{d\mu_1}\right)\mu_2+\left(\frac{d\phi}{d\nu_1}\right)\nu_2=0, \qquad (212)$$

which is the required condition, and may be written

$$A\lambda_1\lambda_2+B\mu_1\mu_2+C\nu_1\nu_2+D(\mu_1\nu_2+\mu_2\nu_1)+E(\nu_1\lambda_2+\nu_2\lambda_1)+F(\lambda_1\mu_2+\lambda_2\mu_1)=0. \qquad (213)$$

159. *To find the condition that the straight lines* $(l_1, m_1, n_1)(l_2, m_2 n_2)$ *may be parallel to conjugate diameters of the conic* $\phi(a, \beta, \gamma) = 0$.

Let $(\lambda_1, \mu_1, \nu_1)$ be the direction-cosines of (l_1, m_1, n_1): then, if (l_2, m_2, n_2) be conjugate to it, its equation (Art. 157. (205).) must be

$$\left(\frac{d\phi}{d\lambda_1}\right)a+\left(\frac{d\phi}{d\mu_1}\right)\beta+\left(\frac{d\phi}{d\nu_1}\right)\gamma=0.$$

We must have, therefore,

$$\frac{\left(\frac{d\phi}{d\lambda_1}\right)}{l_2} = \frac{\left(\frac{d\phi}{d\mu_1}\right)}{m_2} = \frac{\left(\frac{d\phi}{d\nu_1}\right)}{n_2} = -k \text{ (suppose)}.$$

Hence,
$$A\lambda_1 + F\mu_1 + E\nu_1 + l_2k = 0,$$
$$F\lambda_1 + B\mu_1 + D\nu_1 + m_2k = 0,$$
$$E\lambda_1 + D\mu_1 + C\nu_1 + n_2k = 0;$$

we also have
$$l_1\lambda_1 + m_1\mu_1 + n_1\nu_1 = 0, \quad \text{(Art. 31. (26).)}.$$

And, eliminating λ_1, μ_1, ν_1 and k between these equations, we get for the required condition

$$\begin{vmatrix} A, & F, & E, & l_2 \\ F, & B, & D, & m_2 \\ E, & D, & C, & n_2 \\ l_1, & m_1, & n_1, & 0 \end{vmatrix} = 0. \qquad (214)$$

If this determinant be expanded it will be found to be

$$A'l_1l_2 + B'm_1m_2 + C'n_1n_2 + D'(m_1n_2 + m_2n_1) + E'(n_1l_2 + n_2l_1) + F'(l_1m_2 + l_2m_1) = 0. \qquad (215)$$

160. *To find the equation of the principal axes of the conic $\phi(\alpha, \beta, \gamma) = 0$.*

The principal axes may be regarded as the locus of points whose polars are perpendicular to the lines joining those points to the centre.

Let $(\alpha_1, \beta_1, \gamma_1)$ be a point on the axes. The polar of $(\alpha_1, \beta_1, \gamma_1)$ has for its equation (Art. 134)

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha + \left(\frac{d\phi}{d\beta_1}\right)\beta + \left(\frac{d\phi}{d\gamma_1}\right)\gamma = 0,$$

and the equation of the right line joining $(\alpha_1, \beta_1, \gamma_1)$ to the centre is

$$\begin{vmatrix}\beta_1, \gamma_1 \\ \bar\beta, \bar\gamma\end{vmatrix}\alpha + \begin{vmatrix}\gamma_1, \alpha_1 \\ \bar\gamma, \bar\alpha\end{vmatrix}\beta + \begin{vmatrix}\alpha_1, \beta_1 \\ \bar\alpha, \bar\beta\end{vmatrix}\gamma = 0;$$

and the condition that these should be perpendicular is (Art. 71. (85).)

$$\left\{\left(\frac{d\phi}{d\alpha_1}\right)-\left(\frac{d\phi}{d\beta_1}\right)\cos C-\left(\frac{d\phi}{d\gamma_1}\right)\cos B\right\}\begin{vmatrix}\beta_1, \gamma_1 \\ \bar\beta, \bar\gamma\end{vmatrix} + \left\{\left(\frac{d\phi}{d\beta_1}\right)\right.$$

$$\left.-\left(\frac{d\phi}{d\gamma_1}\right)\cos A-\left(\frac{d\phi}{d\alpha_1}\right)\cos C\right\}\begin{vmatrix}\gamma_1, \alpha_1 \\ \bar\gamma, \bar\alpha\end{vmatrix} + \left\{\left(\frac{d\phi}{d\gamma_1}\right)-\left(\frac{d\phi}{d\alpha_1}\right)\cos B\right.$$

$$\left.-\left(\frac{d\phi}{d\beta_1}\right)\cos A\right\}\begin{vmatrix}\alpha_1, \beta_1 \\ \bar\alpha, \bar\beta\end{vmatrix} = 0; \tag{216}$$

whence, suppressing the suffixes and using the determinant form, we have for the equation of the locus

$$\begin{vmatrix}\left(\frac{d\phi}{d\alpha}\right)-\left(\frac{d\phi}{d\beta}\right)\cos C-\left(\frac{d\phi}{d\gamma}\right)\cos B, & \alpha, & \bar\alpha, \\ \left(\frac{d\phi}{d\beta}\right)-\left(\frac{d\phi}{d\gamma}\right)\cos A-\left(\frac{d\phi}{d\alpha}\right)\cos C, & \beta, & \bar\beta, \\ \left(\frac{d\phi}{d\gamma}\right)-\left(\frac{d\phi}{d\alpha}\right)\cos B-\left(\frac{d\phi}{d\beta}\right)\cos A, & \gamma, & \bar\gamma,\end{vmatrix} = 0. \tag{217}$$

161. The equation of the axes may be obtained directly in the form of a determinant, as follows.

Let (l, m, n) be an axis and let $(\alpha_1, \beta_1, \gamma_1)$ be a point on it; then, since the centre $(\bar\alpha, \bar\beta, \bar\gamma)$ also lies upon it, we have

$$l\alpha_1 + m\beta_1 + n\gamma_1 = 0, \tag{218}$$

and

$$l\bar\alpha + m\bar\beta + n\bar\gamma = 0. \tag{219}$$

If (l, m, n) be perpendicular to the polar of $(\alpha_1, \beta_1, \gamma_1)$, whose equation is

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha + \left(\frac{d\phi}{d\beta_1}\right)\beta + \left(\frac{d\phi}{d\gamma_1}\right)\gamma = 0,$$

we must have (Arts. 69. 42)

$$l\left\{\left(\frac{d\phi}{d\alpha_1}\right)-\left(\frac{d\phi}{d\beta_1}\right)\cos C-\left(\frac{d\phi}{d\gamma_1}\right)\cos B\right\}+m\left\{\left(\frac{d\phi}{d\beta_1}\right)-\left(\frac{d\phi}{d\gamma_1}\right)\cos A\right.$$

$$\left.-\left(\frac{d\phi}{d\alpha_1}\right)\cos C\right\}+n\left\{\left(\frac{d\phi}{d\gamma_1}\right)-\left(\frac{d\phi}{d\alpha_1}\right)\cos B-\left(\frac{d\phi}{d\beta_1}\right)\cos A\right\}=0; \tag{220}$$

and, eliminating l, m, n between (218), (219), and (220), and suppressing the suffixes, we get for the equation of the locus

$$\begin{vmatrix} \alpha, & \bar{\alpha}, & \left(\frac{d\phi}{d\alpha}\right)-\left(\frac{d\phi}{d\beta}\right)\cos C-\left(\frac{d\phi}{d\gamma}\right)\cos B \\ \beta, & \bar{\beta}, & \left(\frac{d\phi}{d\beta}\right)-\left(\frac{d\phi}{d\gamma}\right)\cos A-\left(\frac{d\phi}{d\alpha}\right)\cos C \\ \gamma, & \bar{\gamma}, & \left(\frac{d\phi}{d\gamma}\right)-\left(\frac{d\phi}{d\alpha}\right)\cos B-\left(\frac{d\phi}{d\beta}\right)\cos A \end{vmatrix}=0. \tag{221}$$

162. *To find the lengths of the semi-axes of the conic* $\phi(\alpha, \beta, \gamma) = 0$.

The length of the central radius, in any direction (λ, μ, ν), is given (Art. 149. (192).) by the equation

$$\frac{\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})}{r^2}+\phi(\lambda, \mu, \nu)=0,$$

or, since (Art. 138. (172).) $\quad \phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = -\dfrac{4S^2\Delta}{{}^{a}\Delta^{b}_{c}}$,

by

$$\frac{\Delta}{{}^{a}\Delta^{b}_{c}}\cdot\frac{4S^2}{r^2}=\phi(\lambda, \mu, \nu); \tag{222}$$

and we have to make $\dfrac{1}{r^2}$ a maximum or minimum; λ, μ, ν being connected by the relations (Art. 37),

$$a\cos A\lambda^2+b\cos B\mu^2+c\cos C\nu^2=\frac{4S^2}{abc}, \tag{223}$$

$$a\lambda+b\mu+c\nu=0. \tag{224}$$

Let then

$$U=(A\lambda^2+B\mu^2+C\nu^2+2D\mu\nu+2E\nu\lambda+2F\lambda\mu)-P(a\cos A\lambda^2$$
$$+b\cos B\mu^2+c\cos C\nu^2)+2Q(a\lambda+b\mu+c\nu),$$

where P and Q are indeterminate multipliers; differentiating we have

$$\frac{1}{2}DU=0=\{(A-Pa\cos A)\lambda+F\mu+E\nu+aQ\}d\lambda+\{(B-Pb\cos B)\mu$$
$$+D\nu+F\lambda+bQ\}d\mu+\{(C-Pc\cos C)\nu+E\lambda+D\mu+cQ\}d\nu;$$

and therefore, simultaneously,

$$\begin{aligned}(A-Pa\cos A)\lambda+F\mu+E\nu+aQ&=0,\\ F\lambda+(B-Pb\cos B)\mu+D\nu+bQ&=0,\\ E\lambda+D\mu+(C-Pc\cos C)\nu+cQ&=0.\end{aligned}\tag{225}$$

Multiplying the equations (225) by λ, μ, ν respectively and adding, we get, by means of (223) and (224),

$$\frac{4PS^2}{abc}=\phi(\lambda, \mu, \nu).\tag{226}$$

Hence, by (222),
$$P=\frac{\Delta}{{}^{a}\underset{c}{\Delta}{}^{b}}\cdot\frac{abc}{r^2},\tag{227}$$

and, substituting this value for P and eliminating λ, μ, ν and Q from (225) and (224), we get

$$\begin{vmatrix} A-\frac{\Delta}{{}^{a}\underset{c}{\Delta}{}^{b}}\cdot\frac{abc}{r^2}a\cos A, & F, & E, & a\\ F, & B-\frac{\Delta}{{}^{a}\underset{c}{\Delta}{}^{b}}\cdot\frac{abc}{r^2}b\cos B, & D, & b\\ E, & D, & C-\frac{\Delta}{{}^{a}\underset{c}{\Delta}{}^{b}}\cdot\frac{abc}{r^2}c\cos C, & c\\ a, & b, & c, & 0\end{vmatrix}=0,\tag{228}$$

a quadratic in $\frac{1}{r^2}$ which gives the lengths of the principal semi-diameters.

163. *To find the area of the conic* $\phi(\alpha, \beta, \gamma) = 0$.

Mr. Ferrers has deduced the area of the conic from the equation (228) of the last Article. If we expand the determinant, and, for convenience, write (A), (B), (C), 0 for its diagonal elements, this quadratic becomes

$$(A)\begin{vmatrix} (B), & D, & b \\ D, & (C), & c \\ b, & c, & 0 \end{vmatrix} - F\begin{vmatrix} F, & E, & a \\ D, & (C), & c \\ b, & c, & 0 \end{vmatrix} + E\begin{vmatrix} F, & E, & a \\ (B), & D, & b \\ b, & c, & 0 \end{vmatrix} - a\begin{vmatrix} F, & E, & a \\ (B), & D, & b \\ D, & (C), & c \end{vmatrix} = 0.$$

or,

$$\begin{gathered}(A)\{-(B)c^2+2Dbc-(C)b^2\}-F\{-Fc^2+Dca+Ebc-(C)ab\} \\ +E\{-Fbc+(B)ca+Eb^2-Dab\}-a\{FDc-(C)Fb \\ +(B)(C)a-(B)Ec+DEb-D^2a\}=0;\end{gathered}$$

or, if we collect only the constant terms and those which involve $\frac{1}{r^4}$, and make the necessary reductions,

$$\begin{gathered}-(A'a^2+B'b^2+C'c^2+2D'bc+2E'ca+2F'ab)+\text{etc.}\ldots \\ -\left(\frac{\Delta\, abc}{\underset{c}{{}^{a}\Delta^{b}}}\right)^2 \cdot \frac{16\, S^2}{r^2} = 0;\end{gathered}$$

that is (Prelim. chap. (E).),

$$\underset{c}{{}^{a}\Delta^{b}} + \text{etc.} \cdots - \left(2\, abc\, S\frac{\Delta}{\underset{c}{{}^{a}\Delta^{b}}}\right)^2 \frac{1}{r^4} = 0. \tag{229}$$

Hence if $\frac{1}{\rho_1^2}$, $\frac{1}{\rho_2^2}$ be the roots of this quadratic, we have;—

$$\begin{aligned}\text{Area of the conic } &= \pi\rho_1\rho_2 \\ &= -\frac{2\pi a\, abc\, S\, \Delta}{\left[\underset{c}{{}^{a}\Delta^{b}}\right]^{\frac{3}{2}}}.\end{aligned} \tag{230}$$

CHAPTER VII.

INTERPRETATION OF PARTICULAR FORMS OF THE EQUATION OF THE SECOND DEGREE.

164. The present chapter will be devoted to the consideration of particular forms of the trilinear equation of the second order. We commence with those which occur in the subjoined list:

$$S_1 - kS_2 = 0, \quad (A)$$

$$S_1 - ktu = 0, \quad (B)$$

$$S_1 - ku^2 = 0, \quad (C)$$

$$S_1 - ku = 0, \quad (D)$$

$$S_1 - k^2 = 0, \quad (E)$$

$$vw - ktu = 0, \quad (F)$$

$$vw - ku^2 = 0, \quad (G)$$

$$vw - ku = 0, \quad (H)$$

$$vw - k^2 = 0. \quad (J)$$

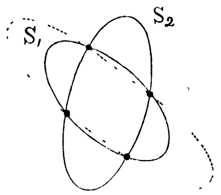
In these, k is any constant and $S_1 = 0$, $S_2 = 0$ represent any two conics; $t = 0$, $u = 0$, $v = 0$, $w = 0$ are the abridged forms of the homogeneous equations of four straight lines (l_1, m_1, n_1), (l_2, m_2, n_2), ... in which the co-efficients may have any possible values.

165. Of the above forms, which will be examined in order, the second, third, fourth and fifth are successively derivable from the first, and the remainder from these, in a manner which will be understood as we proceed.

$$(A).\qquad S_1 - kS_2 = 0$$

is evidently satisfied by any values of α, β, γ which satisfy $S_1 = 0$ and $S_2 = 0$ simultaneously. It therefore represents a conic (or, if k varies, a system of conics) passing through the four points of intersection, whether real, coincident, or imaginary, of S_1 and S_2. Such a curve is indicated by the dotted line in the figure.

Fig. 20.



166. If the conditions of the last Chapter (Art. 125) be applied to this equation, it will appear that it represents a hyperbola, an ellipse, or a parabola, according as

$$\begin{vmatrix} A_1 - kA_2, & F_1 - kF_2, & E_1 - kE_2, & a \\ F_1 - kF_2, & B_1 - kB_2, & D_1 - kD_2, & b \\ E_1 - kE_2, & D_1 - kD_2, & C_1 - kC_2, & c \\ a, & b, & c, & 0 \end{vmatrix}^{\text{a}}$$

is *positive*, *zero*, or *negative*. Its locus will be a pair of right lines (Art. 146) if

$$\begin{vmatrix} A_1 - kA_2, & F_1 - kF_2, & E_1 - kE_2 \\ F_1 - kF_2, & B_1 - kB_2, & D_1 - kD_2 \\ E_1 - kE_2, & D_1 - kD_2, & C_1 - kC_2 \end{vmatrix} = 0. \qquad (231)$$

In like manner the condition for a circular locus (Art. 149) is easily applied.

167. The cubic (231), when the determinant is expanded (Prelim. chap. (D),) becomes

$$(A_1B_1C_1) - \{(A_2B_1C_1) + (A_1B_2C_1) + A_1B_1C_2)\}k + \{(A_1B_2C_2) + (A_2B_1C_2) + (A_2B_2C_1)\}k^2 - (A_2B_2C_2)k^3 = 0. \qquad (232)$$

---

[a] It is assumed that

$$S_1 = A_1\alpha^2 + B_1\beta^2 + C_1\gamma^2 + 2D_1\beta\gamma + 2E_1\gamma\alpha + 2F_1\alpha\beta = 0,$$

and $S_2 = A_2\alpha^2 + B_2\beta^2 + C_2\gamma^2 + 2D_2\beta\gamma + 2E_2\gamma\alpha + 2F_2\alpha\beta = 0.$

Let k_1, k_2, k_3 be the roots of this equation; then it is manifest that

$$\begin{aligned} S_1 - k_1 S_2 &= 0, \\ S_1 - k_2 S_2 &= 0, \\ S_1 - k_3 S_2 &= 0, \end{aligned} \qquad (233)$$

are the equations of the three pairs of chords of intersection of the conics $S_1 = 0$, $S_2 = 0$, and, therefore, of the whole system represented by the equation

$$S_1 - kS_2 = 0.$$

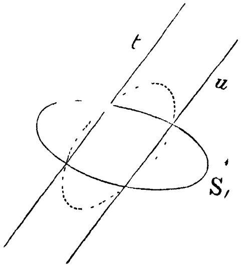
168. Now all this will be true if either S_1 or S_2, or both, be resolvable into linear factors; that is to say, if one or both of the conics degenerate into a pair of right lines. Should a factor be of the form $a\alpha + b\beta + c\gamma$, or, which is the same thing (Art. 3. (1).), a constant quantity, one of the right lines will be at an infinite distance (Art. 52). If the factors be identical, the pair of lines will be coincident.

169. First suppose the conic S_2 to consist of two right lines (l_1, m_1, n_1), (l_2, m_2, n_2), whose equations in their abridged forms are $t = 0$, $u = 0$; the above equation becomes

$$(B). \qquad S_1 - ktu = 0,$$

which, therefore, represents a system of conics passing through the four points of intersection of these two right lines with the conic S_1 (fig. 21). In this case t and u are two of the common chords; one root, therefore, of the cubic (232) will be infinite, and, as we should expect (Art. 146), $(A_2B_2C_2) = 0$.

Fig. 21.

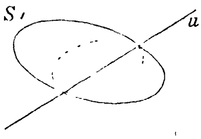


170. Next let the two right lines of which S_2 is composed, coincide. Making $t = u$, we have

(*C*). $$S_1 - ku^2 = 0,$$

whose locus is, therefore, a system of conics passing the two pairs of coincident points in which u meets the conic S_1, and, therefore, having double contact with S_1 at the extremities of the chord $u = 0$ (fig. 22).

Fig. 22.

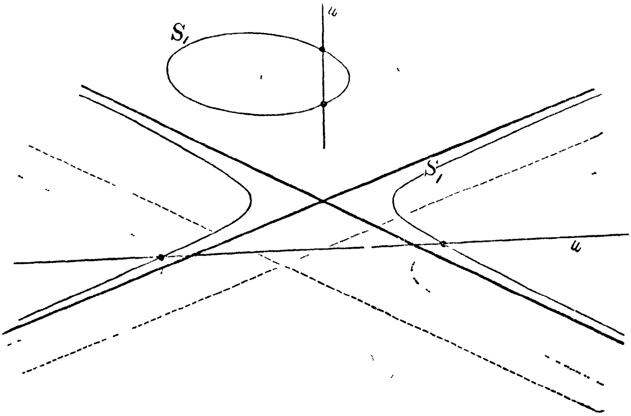


171. It is obvious, from an inspection of the figure, that two out of the three pairs of chords common to S_1, S_2 (Art. 167. (233).) have now come into coincidence with the chord of contact u, and the remaining pair become tangents at its extremities. The cubic (232) will have two infinite roots, and therefore both $(A_2B_2C_2) = 0$ and $(A_1B_2C_2) + (A_2B_1C_2) + (A_2B_2C_1) = 0$: the latter being true independently of the values of A_1, B_1, C_1, etc. . . ., we must have the first minors of the determinant $(A_2B_2C_2)$ all $= 0$; that is $B_2C_2 - D_2^2 = 0$, $C_2A_2 - E_2^2 = 0$, $A_2B_2 - F_2^2 = 0$, $E_2F_2 - A_2D_2 = 0$, etc. . . . (whereof the three latter relations are involved in the three former), or $D_2^2 = B_2C_2$, $E_2^2 = C_2A_2$, $F_2^2 = A_2B_2$, conditions which are evidently satisfied when S_2 is the square of a linear expression.

(*D*). $$S_1 - ku = 0$$

is of a form such as we should derive from (*B*) if we were to

Fig 23.



replace t by a constant quantity. Its locus, therefore, (fig. 23) is a system of conics which meet S_1 where it is cut by u and the line at infinity. This system will be similar and similarly situated with respect to S_1, and the asymptotes of the conics of the system, whether real or imaginary, will be parallel.

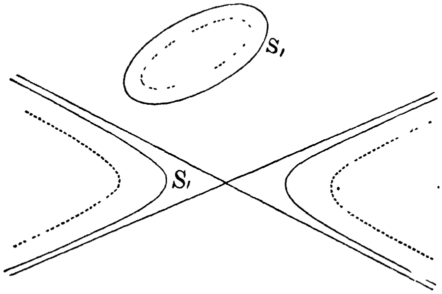
If the curves be parabolas, since the two points at infinity in this case coincide, they will have contact with each other at infinity.

172. The equation

$$(E). \qquad S_1 - k^2 = 0,$$

again, is obviously a particular case of (C), from which it is derivable by the substitution of a constant for u. Hence it denotes a conic (or system of conics) having double contact with S_1, where that curve is met by the straight line at infinity. This system of conics will not only pass through two common points at infinity, but will have common tangents at those points. All the curves of the system, therefore, have the same asymptotes, and are not only similar and similarly situated, but likewise concentric (fig. 24). If the curves be parabolas, they will be equal and have with each other a contact of the third order at infinity.

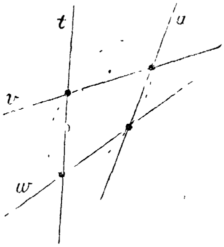
Fig. 24.



173. Now let us suppose the conic S_1 also to degenerate into a pair of straight lines; (E) will become of the form

(*F*). $$vw - ltu = 0,$$

Fig. 25.



which therefore represents a system of conics circumscribing the quadrilateral of which $t = 0$, $u = 0$, and $v = 0$, $w = 0$, are the pairs of opposite sides. The truth of this, however, may be seen without reference to the preceding equations, since (*F*) is evidently satisfied by any one of the suppositions,

$$v = 0, \text{ and } t = 0;$$

$$v = 0, \text{ and } u = 0;$$

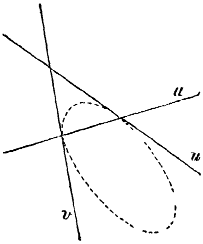
$$w = 0, \text{ and } t = 0;$$

or, $$w = 0, \text{ and } u = 0.$$

174. Similarly, from a comparison of the next equation with (*C*), it appears that

(*G*). $$vw - ku^2 = 0$$

Fig. 26.



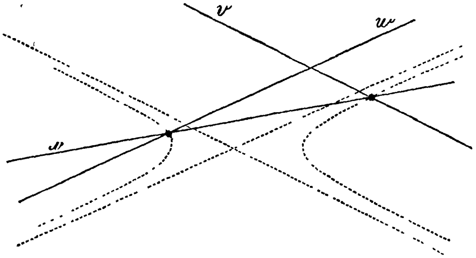
represents a system of conics which have contact at two fixed points; $u = 0$ (fig. 26) being their chord of contact, and $v = 0$, $w = 0$ the tangents at its extremities.

175. Again, by a reference to Art. 171. (*D*), it will be seen that

(*H*). $$vw - ku = 0$$

has for its locus a system of conics having that portion of u which is intercepted between v and w for a common chord, and passing through the two fixed points at infinity in which the line at infinity is met by $v = 0$, $w = 0$. These conics are therefore similar and similarly situated hyperbolas having their asymptotes parallel to $v = 0$, $w = 0$ (fig. 27).

Fig. 27.

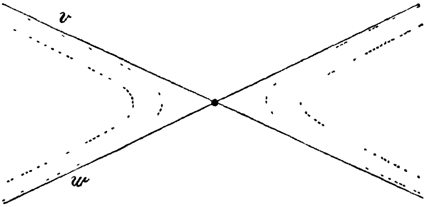


176. The locus of

$$(J). \qquad vw - k^2 = 0$$

is a system of concentric, similar, and similarly situated hyperbolas, having $v = 0$, $w = 0$ for their common asymptotes and the intersection of those lines for a common centre, (fig. 28).

Fig. 28.



177. It is obvious that in the four last cases $v = 0$, $w = 0$, taken together, form one of the family of curves, just as $S_1 = 0$ did in the preceding instances.

178. The meaning of the equations

$$(K). \qquad S_1 + k\alpha\beta = 0,$$

$$(L). \qquad S_1 + k\alpha^2 = 0,$$

$$(M). \qquad S_1 + k\alpha = 0,$$

follows at once from the interpretation just given of equations (B), (C), and (D); the two sides, $\alpha = 0$, $\beta = 0$, of the fundamental triangle taking the place of the two straight lines $t = 0$ and $u = 0$, or $l_1\alpha + m_1\beta + n_1\gamma = 0$ and $l_2\alpha + m_2\beta + n_2\gamma = 0$.

179. We now proceed to consider some of the more special forms of the equation $\phi(\alpha, \beta, \gamma) = 0$ which we employed in the last chapter, and to indicate the nature of their loci and the relation in which they stand to the triangle of reference. A number of typical forms are collected here for the sake of reference, and will be discussed in the order in which they occur.

$$L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0, \qquad (N)$$

$$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 \pm 2MN\beta\gamma \pm 2NL\lambda\alpha \pm 2LM\alpha\beta = 0, \qquad (O)$$

$$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 = 0, \qquad (P)$$

$$L^2\alpha^2 - M^2\beta^2 - N^2\gamma^2 = 0, \qquad (Q)$$

$$\beta\gamma - k\alpha^2 = 0, \qquad (R)$$

$$\beta\gamma - k\alpha = 0, \qquad (S)$$

$$\beta\gamma - k^2 = 0, \qquad (T)$$

$$\beta^2 - k\alpha = 0. \qquad (U)$$

180. The first equation

$$(N). \qquad L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0,$$

is satisfied by any one of the three following suppositions,

$$\beta = 0 \text{ and } \gamma = 0;$$

$$\gamma = 0 \text{ and } \alpha = 0;$$

or, $$\alpha = 0 \text{ and } \beta = 0.$$

The conic, therefore, which it represents passes through the three angular points of the triangle of reference.

181. The equation may be written in the form

$$L\beta\gamma + (M\gamma + N\beta)\alpha = 0,$$

hence (Art. 173. (*F*).) $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and $M\gamma + N\beta = 0$, are the four sides of an inscribed quadrilateral; and, therefore, since $\beta = 0$, $\gamma = 0$ intersect on the conic (Art. 180) the fourth side is evidently a tangent to the curve at A.

Hence

$$\frac{\beta}{M} + \frac{\gamma}{N} = 0,$$

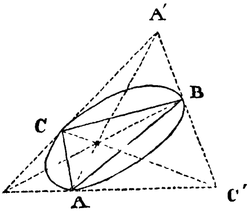
$$\frac{\gamma}{N} + \frac{\alpha}{L} = 0, \qquad (234)$$

$$\frac{\alpha}{L} + \frac{\beta}{M} = 0,$$

are the equations of the tangents to the circumscribing conic at A, B, C respectively. These tangents evidently meet the sides of the triangle which respectively subtend their points of contact in points which lie on the same right line

$$\frac{\alpha}{L} + \frac{\beta}{M} + \frac{\gamma}{N} = 0. \qquad (235)$$

Fig. 29.



For this right line, since the equation may be written in the form $\frac{\alpha}{L} + \left(\frac{\beta}{M} + \frac{\gamma}{N}\right) = 0$, must pass through the point of intersection of the tangent at A with the side BC; and similarly for the other two tangents.

182. If $A'B'C'$ be the triangle formed by the three tangents at the angular points of the triangle of reference, the equations of AA', BB', CC' are easily shewn to be respectively

$$\frac{\beta}{M} - \frac{\gamma}{N} = 0,$$

$$\frac{\gamma}{N} - \frac{\alpha}{L} = 0, \qquad (236)$$

$$\frac{\alpha}{L} - \frac{\beta}{M} = 0:$$

and since these equations when added together vanish identically we conclude (Art. 17) that *AA', BB', CC' meet in a point.*

183. If $(\alpha_0, \beta_0, \gamma_0)$ be the co-ordinates of this point of intersection, we shall have

$$\frac{\alpha_0}{L} = \frac{\beta_0}{M} = \frac{\gamma_0}{N} = \frac{2S}{aL + bM + cN}. \qquad (237)$$

184. From the form of equations (234) and (236) it appears (Art. 95) that *any side of the triangle $A'B'C'$, and the straight line which joins its point of contact to the opposite vertex, are harmonic conjugates with respect to the two sides of the triangle of reference which meet in that point of contact.*

185. With regard to the nature of the curve whose equation we are considering it is to be observed, that, since in this case

$$\phi(\alpha, \beta, \gamma) = 0\alpha^2 + 0\beta^2 + 0\gamma^2 + 2L\beta\gamma + 2M\gamma\alpha + 2N\alpha\beta = 0,$$

and, consequently,

$$\phi(a, b, c)' = -L^2a^2 - M^2\beta^2 - N^2\gamma^2 + 2MNbc + 2NLca + 2LMab,$$

the locus of (N) *will be* (Art. 125) *a hyperbola, a parabola, or an ellipse, according as*

$$L^2a^2 + M^2b^2 + N^2c^2 - 2MNbc - 2NLca - 2LMab$$

is positive, zero, or negative.

186.

$$(O). \quad L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 \pm 2MN\beta\gamma \pm 2NL\gamma\alpha \pm 2LM\alpha\beta = 0.$$

The various possible combinations of signs in this equation give us eight distinct forms; *four of these, viz.,*

$$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 + 2MN\beta\gamma + 2NL\gamma\alpha + 2LM\alpha\beta = 0, \quad (O_1)$$

$$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 + 2MN\beta\gamma - 2NL\gamma\alpha - 2LM\alpha\beta = 0, \quad (O_2)$$

$$L^2\alpha^2 + M^2\beta^2 + N\gamma^2 - 2MN\beta\gamma + 2NL\gamma\alpha - 2LM\alpha\beta = 0, \quad (O_3)$$

$$L^2\alpha^2 + M^2\beta^2 + N\gamma^2 - 2MN\beta\gamma - 2NL\gamma\alpha + 2LM\alpha\beta = 0, \quad (O_4)$$

may be written

$$(L\alpha + M\beta + N\gamma)^2 = 0,$$

$$(-L\alpha + M\beta + N\gamma)^2 = 0,$$

$$(L\alpha - M\beta + N\gamma)^2 = 0,$$

and

$$(L\alpha + M\beta - N\gamma)^2 = 0,$$

respectively, and therefore represent pairs of coincident right lines.

The remaining four, viz.,

$$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 - 2MN\beta\gamma - 2NL\gamma\alpha - 2LM\alpha\beta = 0, \quad (O_5)$$

$$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 - 2MN\beta\gamma + 2NL\gamma\alpha + 2LM\alpha\beta = 0, \quad (O_6)$$

$$L^2\alpha^2+M^2\beta^2+N^2\gamma^2+2MN\beta\gamma-2NL\gamma\alpha+2LM\alpha\beta = 0, \qquad (O_7)$$

$$L^2\alpha^2+M^2\beta^2+N^2\gamma^2+2MN\beta\gamma+2NL\gamma\alpha-2LM\alpha\beta = 0, \qquad (O_8)$$

may also be respectively written in the forms

$$(L\alpha)^{\frac{1}{2}} + (M\beta)^{\frac{1}{2}} + (N\gamma)^{\frac{1}{2}} = 0,$$

$$(-L\alpha)^{\frac{1}{2}} + (M\beta)^{\frac{1}{2}} + (N\gamma)^{\frac{1}{2}} = 0,$$

$$(L\alpha)^{\frac{1}{2}} + (-M\beta)^{\frac{1}{2}} + (N\gamma)^{\frac{1}{2}} = 0,$$

$$(L\alpha)^{\frac{1}{2}} + (M\beta)^{\frac{1}{2}} + (-N\gamma)^{\frac{1}{2}} = 0,$$

and represent, respectively, a conic which is inscribed in the triangle of reference, and three other conics which touch one side and the other two sides produced.

187. For the first equation of the last four, viz.,

$$(O_5). \quad L^2\alpha^2+M^2\beta^2+N^2\gamma^2-2MN\beta\gamma-2NL\gamma\alpha-2LM\alpha\beta = 0,$$

$$\text{or,} \qquad (L\alpha)^{\frac{1}{2}} + (M\beta)^{\frac{1}{2}} + (N\gamma)^{\frac{1}{2}} = 0,$$

may be written in the form

$$(M\beta-N\gamma)^2 + L\alpha(L\alpha-2M\beta-2N\gamma) = 0\,;$$

hence (Art. 174. (G).), $\alpha = 0$ and

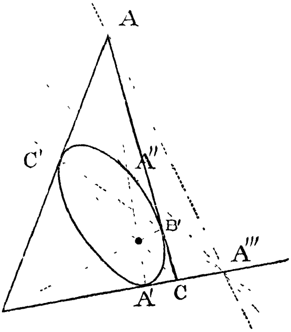
$$L\alpha-2M\beta-2N\gamma = 0$$

are tangents to the conic, and

$$M\beta - N\gamma = 0$$

is their chord of contact. Also, since this chord of contact evidently passes through the vertex A (Art. 16), it appears that, *if the* *points of contact A', B', C', of the inscribed conic be joined to the opposite vertices of the triangle of reference, the equations of the joining lines AA', BB', CC' will be*

Fig. 30.



$$\begin{aligned} M\beta - N\gamma &= 0, \\ N\gamma - La &= 0, \\ La - M\beta &= 0; \end{aligned} \tag{238}$$

they therefore meet in a point (a_0, β_0, γ_0), *such that*

$$La_0 = M\beta_0 = N\gamma_0 = \frac{2S}{\frac{a}{L} + \frac{b}{M} + \frac{c}{N}}. \tag{239}$$

188. Also *the tangents at A'', B'', C'', where the chords meet the conic again have for their equations, respectively,*

$$\begin{aligned} La - 2M\beta - 2N\gamma &= 0, \\ M\beta - 2N\gamma - 2La &= 0, \\ N\gamma - 2La - 2M\beta &= 0. \end{aligned} \tag{240}$$

From the form of these equations it appears that

$$\begin{aligned} M\beta + N\gamma &= 0, \\ N\gamma + La &= 0, \\ La + M\beta &= 0, \end{aligned} \tag{241}$$

are the equations of the straight lines which join the vertices of the triangle of reference to A''', B'''. C''', the points in which the opposite sides are met by the tangents (240).

189. Also, *the points A''', B''', C''' lie in the same straight line*

$$L\alpha + M\beta + N\gamma = 0\,; \tag{242}$$

for the straight line represented by this equation manifestly passes through the intersections of (241) with $\alpha = 0$, $\beta = 0$, $\gamma = 0$, respectively.

190. It will be observed moreover that *AA', AA''' are harmonic conjugates with respect to AB, AC* (Art. 95); and that B and C are the centres of similar harmonic pencils.

191. Again, since the equation (O_5) may be thrown into the form

$$(-L\alpha + M\beta + N\gamma)^2 - 4MN\beta\gamma = 0,$$

we see (Art. 174. (G).) that the equation of $B'C'$, which is the polar of A, is

$$-L\alpha + M\beta + N\gamma = 0\,;$$

similarly,

$$L\alpha - M\beta + N\gamma = 0, \tag{243}$$

and

$$L\alpha + M\beta - N\gamma = 0,$$

are the equations of $C'A'$, $A'B'$, respectively: and from the form of their equations we conclude that these polars, *the sides of the triangle $A'B'C'$, pass through the points A''', B''', C''' respectively.*

192. And further, since the first equations of (241) and (243) may be written in the form

$$-L\alpha + (L\alpha - 2M\beta - 2N\gamma) = 0$$

and

$$-L\alpha - (L\alpha - 2M\beta - 2N\gamma) = 0$$

respectively, it follows (Art. 95) that $\{A''' \,.\, AC', A''A'\}$ *is a harmonic pencil.* In like manner, B''', C''' are respectively the centres of the harmonic pencils $\{B''.BA', B'B'\}$ and $\{C'''.CB', C''C'\}$.

193. When

$$\phi(\alpha, \beta, \gamma) = L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 - 2MN\beta\gamma - 2NL\gamma\alpha - 2LM\alpha\beta = 0,$$

$$\phi(a, b, c)' = 0a^2 + 0b^2 + 0c^2 + 4L^2MNbc + 4LM^2Nca + 4LMN^2ab;$$

therefore (Art. 125) *the inscribed conic represented by the equation* (O_5) *will be an ellipse, a parabola, or hyperbola, according as*

$$Lbc + Mca + Nab$$

or

$$\frac{L}{a} + \frac{M}{b} + \frac{N}{c}$$

is positive, zero, or negative.

The equations of the escribed conics (O_6), (O_7), (O_8) (Art. 186) may be discussed in a similar manner.

194.

(*P*). $$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 = 0.$$

This equation, since the terms are all essentially positive, can be satisfied by no real and possible values of the variables, and therefore represents only an imaginary locus.

If, however, two of the terms only have like signs, the equation is of the form

(*Q*). $$L^2\alpha^2 - M^2\beta^2 - N^2\gamma^2 = 0;$$

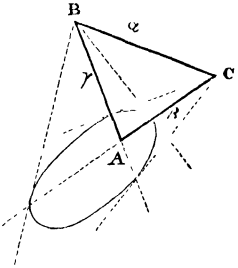
and the conic which it represents stands in an important relation to the triangle of reference. This equation we shall now discuss, and whenever hereafter, for the sake of symmetry, the equation (*P*) is employed, it must be understood that one of the three quantities *L*, *M*, *N* is to be regarded as imaginary.

195. The equation (Q) may be written in either of the forms

$$(L\alpha + M\beta)(L\alpha - M\beta) - N^2\gamma^2 = 0,$$

$$(L\alpha + N\gamma)(L\alpha - N\gamma) - M^2\beta^2 = 0,$$

Fig. 31.



and therefore (Art. 174. (G).) represents a conic which is so situated that CA ($\beta = 0$) is the chord of contact of

$$N^2\gamma^2 - L^2\alpha^2 = 0,$$

the pair of tangents through B, and AB ($\gamma = 0$) that of

$$L^2\alpha^2 - M^2\beta^2 = 0,$$

the pair through C (fig. 31).

In other words, B and C are the poles of CA and AB respectively, and, consequently, A is the pole of BC. So that *the equation* (Q) *represents a conic with respect to which the triangle of reference is self-conjugate.*

196. Writing the equation in the form

$$L^2\alpha^2 - (M^2\beta^2 + N^2\gamma^2) = 0,$$

we see (Art. 174. (G).) that, although A is the pole of the opposite sides, the pair of tangents through A are imaginary.

197. It will be observed that *the pair of tangents through any vertex of the conjugate triangle form a harmonic pencil with the sides which meet in that vertex.*

Fig. 31, which shews the position of the conic (Q) with regard to the triangle of reference, will, if the letters be twice interchanged in a circular order, represent successively the loci of the equations

$$-L^2\alpha^2 + M^2\beta^2 - N^2\gamma^2 = 0,$$

$$-L^2\alpha^2 - M^2\beta^2 + N^2\gamma^2 = 0.$$

198. Let

$$L_1^2\alpha^2 - M_1^2\beta^2 - N_1^2\gamma^2 = 0 \qquad (244)$$

and

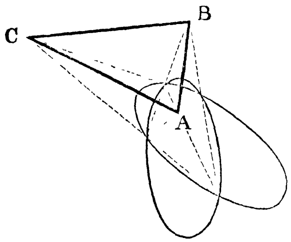
$$L_2^2\alpha^2 - M_2^2\beta^2 - N_2^2\gamma^2 = 0, \qquad (245)$$

be the equations of two conics with regard to which the triangle of reference is self-conjugate. Eliminating α, β, γ successively, we get

$$\begin{aligned}(L_1^2M_2^2 - L_2^2M_1^2)\beta^2 - (N_1^2L_2^2 - N_2^2L_1^2)\gamma^2 &= 0,\\ (M_1^2N_2^2 - M_2^2N_1^2)\gamma^2 + (L_1^2M_2^2 - L_2^2M_1^2)\alpha^2 &= 0, \qquad (246)\\ (N_1^2L_2^2 - N_2^2L_1^2)\alpha^2 + (M_1^2N_2^2 - M_2^2N_1^2)\beta^2 &= 0,\end{aligned}$$

for the equations of the common chords of (244) and (245); these therefore (fig. 32) pass two and two through the three vertices of the triangle.

Fig 32.



199. Hence also, if a system of conics be described through four fixed points, the points in which the three pairs of opposite connectors intersect form a *conjugate triad* with respect to each curve of the system, and the triangle formed by joining them is a *self-conjugate triangle.*

200. In this case

$$\phi(\alpha, \beta, \gamma) = L^2\alpha^2 - M^2\beta - N^2\gamma^2 + 0\beta\gamma + 0\gamma\alpha + 0\alpha\beta = 0;$$

$$\phi(a, b, c)' = M^2N^2a^2 - N^2L^2b^2 - L^2M^2c^2.$$

Wherefore, (Q) *will represent an ellipse, a parabola, or hyperbola, according as*

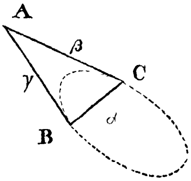
$$\frac{a^2}{L^2} - \frac{b^2}{M^2} - \frac{c^2}{N^2}$$

is positive, zero, or negative.

201.

$$(R). \qquad \beta\gamma - k\alpha^2 = 0.$$

Fig. 33.



By reference to (Art. 174. (*G*).) it will be seen that *the equation* (*R*) *represents a conic section to which* *CA* ($\beta = 0$) *and* *AB* ($\gamma = 0$) *are tangents, while* *BC* ($\alpha = 0$) *is their chord of contact.*

202. Here

$$\phi(\alpha, \beta, \gamma) = 2k\alpha^2 + 0\beta^2 + 0\gamma^2 - 2\beta\gamma + 0\gamma\alpha + 0\alpha\beta = 0;$$

$$\phi(a, b, c)' = -a^2 + 4kbc:$$

and *the locus of* (*R*) *will be an ellipse, a parabola, or hyperbola, according as*

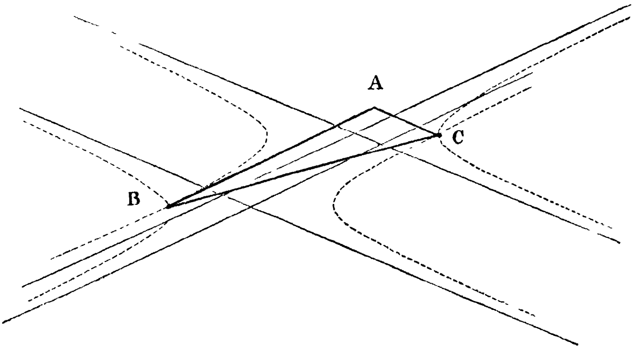
$$k \gtreqless \frac{a^2}{4bc}.$$

203.

$$(S). \qquad \beta\gamma - k\alpha = 0.$$

The locus of this equation (Art. 175. (*H*).) is *a system of similar*

Fig. 34.



and similarly-placed hyperbolas (fig. 34), *having the side BC of the triangle of reference for a common chord, and passing through the points at infinity on the other two sides.* Each curve of the system, therefore, has its asymptotes parallel to CA, AB.

204. *Equation* (S), as we should expect, *satisfies the condition for a huperbola;* for we have

$$\phi(\alpha, \beta, \gamma) = 2a\alpha^2 + 0\beta^2 + 0\gamma^2 - 2\left(\frac{2S}{k}\right)\beta\gamma + 2c\gamma\alpha + 2b\alpha\beta = 0,$$

and therefore

$$\phi(a, b, c)' = -\frac{4S^2}{k^2}a^2 - c^2b^2 - b^2c^2 + 2\left(bc + \frac{4aS}{k}\right)bc - 2\left(\frac{2bS}{k}\right)ca - 2\left(\frac{2cS}{k}\right)ab$$

$$= -\frac{4a^2S^2}{k^2}, \text{ essentially a negative quantity.}$$

205. Again the equation

$$(T). \qquad \beta\gamma - k^2 = 0$$

gives us

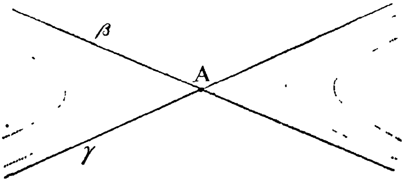
$$\phi(\alpha, \beta, \gamma) = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 + 2\left(bc - \frac{2S^2}{k^2}\right)\beta\gamma + 2ca\gamma\alpha + 2ab\alpha\beta = 0;$$

whence we shall find that

$$\phi(a, b, c)' = -\frac{4a^2S^4}{k^4},$$

and that (Arts. 176. 125) (T) *represents a system of concentric,*

Fig. 35.



similar, and similarly situated hyperbolas, having the two sides CA, AB, of the triangle of reference for their common asymptotes (fig. 35).

206. Lastly, the equation

$$(U). \qquad \beta^2 - k\alpha = 0$$

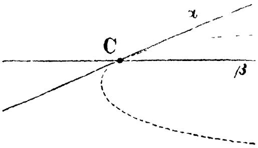
has for its locus a system of conics (Art. 175. (*H*).) to which $\alpha = 0$ and the line at infinity are tangents at the extremities of the common chord $\beta = 0$. We shall now have

$$\phi(\alpha, \beta, \gamma) = 2a\alpha^2 - \frac{4S}{k}\beta^2 + 0\gamma^2 + 0\beta\gamma + 2c\gamma\alpha + 2b\alpha\beta = 0\ ;$$

$$\phi(a, b, c)' = -c^2b^2 - \left(\frac{8aS}{k} + b^2\right)c^2 + 2(bc)bc + 2\left(\frac{4cS}{k}\right)ca,$$
$$= 0\ ;$$

and (*U*) *represents a system of similar and similarly placed parabolas*

Fig. 36.



of which $\beta = 0$ *is a diameter, and* $\alpha = 0$ *the tangent at its extremity* (fig. 36).

CHAPTER VIII.

EQUATIONS OF THE SECOND ORDER CONTINUED.

207. Instances will now be set before the reader of the mode of application to particular equations of the general results of Chapter VI; the proof being, however, given in an independent form where the case appears to deserve a separate investigation. For the convenience of the student some of the most important forms of the equation of the second order are collected in the next Article.

208.

(N). $$L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0,$$

or $$\frac{L}{\alpha} + \frac{M}{\beta} + \frac{N}{\gamma} = 0,$$

which represents a conic circumscribing the triangle of reference (Art. 180).

(O_5). $$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 - 2MN\beta\gamma - 2NL\gamma\alpha - 2LM\alpha\beta = 0,$$

or $$(L\alpha)^{\frac{1}{2}} + (M\beta)^{\frac{1}{2}} + (N\gamma)^{\frac{1}{2}} = 0,$$

which represents a conic inscribed in the triangle of reference (Art. 187).

(P). $$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 = 0,$$

which is the equation of a conic with respect to which the triangle of reference is self-conjugate (see Art. 194. (P). (Q).).

(*R*). $$\beta\gamma - k\alpha^2 = 0,$$

which is the equation of a conic touching the sides CA, AB, of the triangle of reference at the points C and B respectively (Art. 201).

(*S*). $$\beta\gamma - k\alpha = 0,$$

which represents a hyperbola passing through B and C and having its asymptotes parallel to CA, AB (Art. 203).

(*T*). $$\beta\gamma - k^2 = 0,$$

which is the equation of a hyberbola whose centre is at A, and to which the sides CA, AB are asymptotes (Art. 205).

(*U*). $$\beta^2 - k\alpha = 0,$$

which represents a parabola of which CA is a diameter and BC the tangent at its extremity (Art. 206).

To these may be added the equation

(*V*). $$\alpha\gamma - k\beta\delta = 0,$$

which *represents a conic circumscribing a quadrilateral whose sides are* $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, and, though not strictly speaking trilinear, may be regarded as a particular case of (Art. 173. (*F*).).

209. *Equation of the chord joining the two points* $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$.

(*N*). $$\frac{L}{\alpha} + \frac{M}{\beta} + \frac{N}{\gamma} = 0.$$

The equation of the straight line joining the two given points (Art. 58. (68).) is

$$(\beta_1\gamma_2 - \beta_2\gamma_1)\alpha + (\gamma_1\alpha_2 - \gamma_2\alpha_1)\beta + (\alpha_1\beta_2 - \alpha_2\beta_1)\gamma = 0; \quad (247)$$

But, since in this case the points lie on the conic (N), we have

$$\frac{L}{\alpha_1} + \frac{M}{\beta_1} + \frac{N}{\gamma_1} = 0$$

and
$$\frac{L}{\alpha_2} + \frac{M}{\beta_2} + \frac{N}{\gamma_2} = 0;$$

whence,
$$\frac{L}{\frac{1}{\beta_1\gamma_2} - \frac{1}{\beta_2\gamma_1}} = \frac{M}{\frac{1}{\gamma_1\alpha_2} - \frac{1}{\gamma_2\alpha_1}} = \frac{N}{\frac{1}{\alpha_1\beta_2} - \frac{1}{\alpha_2\beta_1}},$$

or
$$\frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\frac{L}{\alpha_1\alpha_2}} = \frac{\gamma_1\alpha_2 - \gamma_2\alpha_1}{\frac{M}{\beta_1\beta_2}} = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\frac{N}{\gamma_1\gamma_2}}; \quad (248)$$

and (247) may be written

$$\frac{L}{\alpha_1\alpha_2}\alpha + \frac{M}{\beta_1\beta_2}\beta + \frac{N}{\gamma_1\gamma_2}\gamma = 0, \quad (249)$$

which is the required equation.

(O_5).
$$L^{\frac{1}{2}}\alpha^{\frac{1}{2}} + M^{\frac{1}{2}}\beta^{\frac{1}{2}} + N^{\frac{1}{2}}\gamma^{\frac{1}{2}} = 0.$$

Since $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ are points on the curve, we have

$$L^{\frac{1}{2}}\alpha_1^{\frac{1}{2}} + M^{\frac{1}{2}}\beta_1^{\frac{1}{2}} + N^{\frac{1}{2}}\gamma_1^{\frac{1}{2}} = 0,$$

and
$$L^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} + M^{\frac{1}{2}}\beta_2^{\frac{1}{2}} + N^{\frac{1}{2}}\gamma_2^{\frac{1}{2}} = 0;$$

whence,
$$\frac{L^{\frac{1}{2}}}{\beta_1^{\frac{1}{2}}\gamma_2^{\frac{1}{2}} - \beta_2^{\frac{1}{2}}\gamma_1^{\frac{1}{2}}} = \frac{M^{\frac{1}{2}}}{\gamma_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} - \gamma_2^{\frac{1}{2}}\alpha_1^{\frac{1}{2}}} = \frac{N^{\frac{1}{2}}}{\alpha_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}} - \alpha_2^{\frac{1}{2}}\beta_1^{\frac{1}{2}}};$$

or,

$$\frac{\beta_1\gamma_2 - \beta_2\gamma_1}{L^{\frac{1}{2}}(\beta_1^{\frac{1}{2}}\gamma_2^{\frac{1}{2}} + \beta_2^{\frac{1}{2}}\gamma_1^{\frac{1}{2}})} = \frac{\gamma_1\alpha_2 - \gamma_2\alpha_1}{M^{\frac{1}{2}}(\gamma_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} + \gamma_2^{\frac{1}{2}}\alpha_1^{\frac{1}{2}})} = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{N^{\frac{1}{2}}(\alpha_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}} + \alpha_2^{\frac{1}{2}}\beta_1^{\frac{1}{2}})}; \tag{250}$$

and (247) becomes

$$L^{\frac{1}{2}}(\beta_1^{\frac{1}{2}}\gamma_2^{\frac{1}{2}} + \beta_2^{\frac{1}{2}}\gamma_1^{\frac{1}{2}})\alpha + M^{\frac{1}{2}}(\gamma_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} + \gamma_2^{\frac{1}{2}}\alpha_1^{\frac{1}{2}})\beta + N^{\frac{1}{2}}(\alpha_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}} + \alpha_2^{\frac{1}{2}}\beta_1^{\frac{1}{2}})\gamma = 0, \tag{251}$$

the equation of the chord.

(*P*). $$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 = 0.$$

Since the given points are on the curve, we have

$$L^2\alpha_1^2 + M^2\beta_1^2 + N^2\gamma_1^2 = 0,$$

and $$L^2\alpha_2^2 + M^2\beta_2^2 + N^2\gamma_2^2 = 0,$$

which give

$$\frac{L^2}{\beta_1^2\gamma_2^2 - \beta_2^2\gamma_1^2} = \frac{M^2}{\gamma_1^2\alpha_2^2 - \gamma_2^2\alpha_1^2} = \frac{N^2}{\alpha_1^2\beta_2^2 - \alpha_2^2\beta_1^2}.$$

Hence, $$\frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\dfrac{L^2}{\beta_1\gamma_2 + \beta_2\gamma_1}} = \frac{\gamma_1\alpha_2 - \gamma_2\alpha_1}{\dfrac{M^2}{\gamma_1\alpha_2 + \gamma_2\alpha_1}} = \frac{\alpha_1\beta_2 + \alpha_2\beta_1}{\dfrac{N^2}{\alpha_1\beta_2 + \alpha_2\beta_1}}; \tag{252}$$

and we have for the equation of the chord,

$$\frac{L^2\alpha}{\beta_1\gamma_2 + \beta_2\gamma_1} + \frac{M^2\beta}{\gamma_1\alpha_2 + \gamma_2\alpha_1} + \frac{N^2\gamma}{\alpha_1\beta_2 + \alpha_2\beta_1} = 0. \tag{253}$$

(*R*). $$\beta\gamma - k\alpha^2 = 0.$$

We have, since the given points are on the curve,

$$\beta_1\gamma_1 = k\alpha_1^2, \tag{254}$$

$$\beta_2\gamma_2 = k\alpha_2^2. \qquad (255)$$

Hence, multiplying (254) and (255) by γ_2^2 and γ_1^2 respectively, subtracting, and dividing by $(\gamma_1\alpha_2 + \gamma_2\alpha_1)$, we get

$$\frac{\beta_1\gamma_2 - \beta_2\gamma_1}{-k(\gamma_1\alpha_2 + \gamma_2\alpha_1)} = \frac{\gamma_1\alpha_2 - \gamma_2\alpha_1}{\gamma_1\gamma_2}. \qquad (256)$$

Similarly, if we multiply (254) and (255) by β_2^2 and β_1^2 and subtract, we get

$$\frac{\beta_1\gamma_2 - \beta_2\gamma_1}{-k(\alpha_1\beta_2 + \alpha_2\beta_1)} = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\beta_1\beta_2}; \qquad (257)$$

and (256) and (257) give

$$\frac{\beta_1\gamma_2 - \beta_2\gamma_1}{-k} = \frac{\gamma_1\alpha_2 - \gamma_2\alpha_1}{\dfrac{\gamma_1\gamma_2}{\gamma_1\alpha_2 + \gamma_2\alpha_1}} = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\dfrac{\beta_1\beta_2}{\alpha_1\beta_2 + \alpha_2\beta_1}}; \qquad (258)$$

whence substituting in (247), we have for the required equation

$$-k\alpha + \frac{\gamma_1\gamma_2}{\gamma_1\alpha_2 + \gamma_2\alpha_1}\beta + \frac{\beta_1\beta_2}{\alpha_1\beta_2 + \alpha_2\beta_1}\gamma = 0. \qquad (259)$$

Second method.—The equation of the chord may, however, be more easily formed, in the present instance, after the manner of Art. 126, as follows. The equation

$$(\beta-\beta_1)(\gamma-\gamma_2) + (\beta-\beta_2)(\gamma-\gamma_1) - 2k(\alpha-\alpha_1)(\alpha-\alpha_2) = 2\beta\gamma - 2k\alpha^2 \qquad (260)$$

must represent a right line, since it is linear; also it is satisfied by the co-ordinates of each of the given points; it must therefore be the equation of the chord.

Equation (260) may be written

$$2k(\alpha_1+\alpha_2)\alpha - (\gamma_1+\gamma_2)\beta - (\beta_1+\beta_2)\gamma - 2k\alpha_1\alpha_2 + \beta_1\gamma_2 + \beta_2\gamma_1 = 0, \qquad (261)$$

which may be readily thrown into the homogeneous form (Art. 4).

(*S*). $$\beta\gamma - k\alpha = 0.$$

Here the chord will be represented by the equation

$$(\beta-\beta_1)(\gamma-\gamma_2) + (\beta-\beta_2)(\gamma-\gamma_1) = 2\beta\gamma-2k\alpha, \qquad (262)$$

or, $$2k\alpha-(\gamma_1+\gamma_2)\beta-(\beta_1+\beta_2)\gamma+(\beta_1\gamma_2+\beta_2\gamma_1) = 0, \qquad (263)$$

for (262) is evidently linear and is satisfied by $(\alpha_1, \beta_1, \gamma_1)$ or by $(\alpha_2, \beta_2, \gamma_2)$. The equation just found may be rendered homogeneous by the method of Art. 4.

(*T*). $$\beta\gamma - k^2 = 0.$$

As in the last section, the equation of the chord of (*T*) is easily seen to be

$$(\beta-\beta_1)(\gamma-\gamma_2) + (\beta-\beta_2)(\gamma-\gamma_1) = 2\beta\gamma-2k^2, \qquad (264)$$

which is also not in the homogeneous form.

(*U*). $$\beta^2 - k\alpha = 0.$$

In this case the linear equation

$$(\beta - \beta_1)(\beta - \beta_2) = \beta^2 - k\alpha \qquad (265)$$

is satisfied by the co-ordinates of the given points on the conic (*U*), and therefore represents the chord which joins them. It may be written

$$k\alpha - (\beta_1 + \beta_2)\beta + \beta_1\beta_2 = 0. \qquad (266)$$

210. *Equation of the tangent at the point* $(\alpha_1, \beta_1, \gamma_1)$.

The equation of the tangent may either be deduced from that of the chord (Art. 209) by making $\alpha_2 = \alpha_1$, $\beta_2 = \beta_1$, $\gamma_2 = \gamma_1$; or be formed after eqq. (152), (153). If the latter method be adopted, it must be remembered that the form (153) may only be used when the equation of the conic is homogeneous, and that in other cases (152), the general equation of the tangent, must be employed.

$$(N). \qquad L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0.$$

Putting $\alpha_2 = \alpha_1$, $\beta_2 = \beta_1$, $\gamma_2 = \gamma_1$ in (249) the equation of the chord, we have

$$\frac{L}{\alpha_1^2}\alpha + \frac{M}{\beta_1^2}\beta + \frac{N}{\gamma_1^2}\gamma = 0, \tag{267}$$

which is the equation of the tangent at $(\alpha_1, \beta_1, \gamma_1)$.

Second method.—The equation of the tangent is (Art. 129. (153).)

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha + \left(\frac{d\phi}{d\beta_1}\right)\beta + \left(\frac{d\phi}{d\gamma_1}\right)\gamma = 0, \tag{268}$$

and, in the present case,

$$\begin{aligned} \left(\frac{d\phi}{d\alpha_1}\right) &= M\gamma_1 + N\beta_1, \\ \left(\frac{d\phi}{d\beta_1}\right) &= N\alpha_1 + L\gamma_1, \\ \left(\frac{d\phi}{d\gamma_1}\right) &= L\beta_1 + M\alpha_1; \end{aligned} \tag{269}$$

hence,

$$(M\gamma_1 + N\beta_1)\alpha + (N\alpha_1 + L\gamma_1)\beta + (L\beta_1 + M\alpha_1)\gamma = 0 \tag{270}$$

is the required equation.

It is obvious that (267) and (270) may be derived the one from the other by means of (N).

(O_5). $$L^{\frac{1}{2}}\alpha^{\frac{1}{2}} + M^{\frac{1}{2}}\beta^{\frac{1}{2}} + N^{\frac{1}{2}}\gamma^{\frac{1}{2}} = 0.$$

Making $\alpha_2 = \alpha_1$, $\beta_2 = \beta_1$, $\gamma_2 = \gamma_1$, in (251) we get for the equation of the tangent at $(\alpha_1, \beta_1, \gamma_1)$

$$\frac{L^{\frac{1}{2}}}{\alpha_1^{\frac{1}{2}}}\alpha + \frac{M^{\frac{1}{2}}}{\beta_1^{\frac{1}{2}}}\beta + \frac{N^{\frac{1}{2}}}{\gamma_1^{\frac{1}{2}}}\gamma = 0. \tag{271}$$

The second method will give the same result, since

$$\left(\frac{d\phi}{d\alpha_1}\right) = \frac{L^{\frac{1}{2}}}{\alpha_1^{\frac{1}{2}}}, \left(\frac{d\phi}{d\beta_1}\right) = \frac{M^{\frac{1}{2}}}{\beta_1^{\frac{1}{2}}}, \left(\frac{d\phi}{d\gamma_1}\right) = \frac{N^{\frac{1}{2}}}{\gamma_1^{\frac{1}{2}}}. \tag{272}$$

If the equation of the conic be taken in the form

$$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 - 2MN\beta\gamma - 2NL\gamma\alpha - 2LM\alpha\beta = 0$$

(153) gives the equation of the tangent in the form

$$L(L\alpha_1 - M\beta_1 - N\gamma_1)\alpha + M(M\beta_1 - N\gamma_1 - L\alpha_1)\beta + N(N\gamma_1 - L\alpha_1 - M\beta_1)\gamma = 0$$
(273

(P). $$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 = 0.$$

The equation of the tangent, obtained by either method, is

$$L^2\alpha_1\alpha + M^2\beta_1\beta + N^2\gamma_1\gamma = 0. \tag{274}$$

(R). $$\beta\gamma - k\alpha^2 = 0.$$

The equation of the tangent, derived from (259) Art. (209), is

$$2k\alpha_1\alpha - \gamma_1\beta - \beta_1\gamma = 0. \tag{275}$$

Also, since

$$\left(\frac{d\phi}{d\alpha_1}\right) = -2k\alpha_1,\ \left(\frac{d\phi}{d\beta_1}\right) = \gamma_1,\ \left(\frac{d\phi}{d\gamma_1}\right) = \beta_1, \tag{276}$$

the equation (268) gives exactly the same result. If the equation (261) be employed, the equation of the tangent will be obtained in the non-homogeneous form

$$2k\alpha_1\alpha - \gamma_1\beta - \beta_1\gamma - k\alpha_1^2 + \beta_1\gamma_1 = 0,$$

which, however, reduces to the form (275) by reason of (R).

(S). $$\beta\gamma - k\alpha = 0.$$

The equation of the chord (Art. 209. (263).) gives for the equation of the tangent at $(\alpha_1, \beta_1, \gamma_1)$

$$k\alpha - \gamma_1\beta - \beta_1\gamma + \beta_1\gamma_1 = 0;$$

or, since by (S) $$\beta_1\gamma_1 = k\alpha_1, \tag{277}$$

$$k(\alpha + \alpha_1) - \gamma_1\beta - \beta_1\gamma = 0, \tag{278}$$

which (Art. 4) may be rendered homogeneous, if necessary.

Second method.—The equation of the tangent at $(\alpha_1, \beta_1, \gamma_1)$, (Art. 129. (152).) is

$$\left(\frac{d\phi}{d\alpha_1}\right)(\alpha-\alpha) + \left(\frac{d\phi}{d\beta_1}\right)(\beta-\beta_1) + \left(\frac{d\phi}{d\gamma_1}\right)(\gamma-\gamma_1) = 0; \tag{279}$$

therefore, since

$$\left(\frac{d\phi}{d\alpha_1}\right) = -k,\ \left(\frac{d\phi}{d\beta_1}\right) = \gamma_1,\ \left(\frac{d\phi}{d\gamma_1}\right) = \beta_1, \tag{280}$$

the required equation is

$$k(\alpha - \alpha_1) - \gamma_1(\beta - \beta_1) - \beta_1(\gamma - \gamma_1) = 0, \tag{281}$$

which, when $\beta_1\gamma_1$ is replaced by $k\alpha_1$ (277), is identical with (278).

(T). $$\beta\gamma - k^2 = 0.$$

In this case $\left(\frac{d\phi}{d\alpha_1}\right) = 0$, $\left(\frac{d\phi}{d\beta_1}\right) = \gamma_1$, $\left(\frac{d\phi}{d\gamma_1}\right) = \beta_1$, and the equation of the tangent (Art. 129. (152).) is

$$\gamma_1(\beta-\beta_1) + \beta_1(\gamma-\gamma_1) = 0, \tag{282}$$

or, since, by (T), $$\beta_1\gamma_1 = k^2,$$

$$\gamma_1\beta + \beta_1\gamma = 2k^2. \tag{283}$$

(U). $$\beta^2 - k\alpha = 0.$$

Here $$\left(\frac{d\phi}{d\alpha_1}\right) = -k,\ \left(\frac{d\phi}{d\beta_1}\right) = 2\beta_1,\ \left(\frac{d\phi}{d\gamma_1}\right) = 0;$$

and (152) becomes

$$-k(\alpha-\alpha_1) + 2\beta_1(\beta-\beta_1) = 0, \tag{284}$$

or, by (U), $$k(\alpha + \alpha_1) - 2\beta_1\beta = 0. \tag{285}$$

211. *Equation of the polar of the point* $(\alpha_1, \beta_1, \gamma_1)$.

From the results of Arts. 134 and 135 it appears that the equation of the polar of the curve $\phi(\alpha, \beta, \gamma) = 0$, *when the equation is homogeneous and of the second degree, is*

$$\left(\frac{d\phi}{d\alpha_1}\right)\alpha + \left(\frac{d\phi}{d\beta_1}\right)\beta + \left(\frac{d\phi}{d\gamma_1}\right)\gamma = 0, \tag{286}$$

identical with the equation of the tangent at the point $(\alpha_1, \beta_1, \gamma_1)$; and is

$$\left(\frac{d\phi}{d\alpha_1}\right)(\alpha-\alpha_1)+\left(\frac{d\phi}{d\beta_1}\right)(\beta-\beta_1)+\left(\frac{d\phi}{d\gamma_1}\right)(\gamma-\gamma_1)+2\phi(\alpha_1,\beta_1,\gamma_1)=0, \quad (287)$$

when the equation of the curve, though not homogeneous, is of the second degree.

Hence the equation of the polar may be readily formed in the case of any of the curves we are considering. It may also be deduced from the equation of the tangent by the method which was applied in the general case (Art. 134). Examples of each method are subjoined.

(*N*). $$L\beta\gamma+M\gamma\alpha+N\alpha\beta=0.$$

Substituting in (286) the values of $\left(\frac{d\phi}{d\alpha_1}\right)$, $\left(\frac{d\phi}{d\beta_1}\right)$, $\left(\frac{d\phi}{d\gamma_1}\right)$ given in (269), we have, for the equation of the polar,

$$(M\gamma_1+N\beta_1)\alpha+(N\alpha_1+L\gamma_1)\beta+(L\beta_1+M\alpha_1)\gamma=0. \quad (288)$$

Second method.—Suppose $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ to be the points of contact of tangents drawn through $(\alpha_1, \beta_1, \gamma_1)$ to the conic. The equations of the tangents (Art. 210. (270).), since $(\alpha_1, \beta_1, \gamma_1)$ lies on each, give

$$(M\gamma_2+N\beta_2)\alpha+(N\alpha_2+L\gamma_2)\beta+(L\beta_2+M\alpha_2)\gamma=0,$$

and $$(M\gamma_3+N\beta_3)\alpha+(N\alpha_3+L\gamma_3)\beta+(L\beta_3+M\alpha_3)\gamma=0.$$

Hence the points of contact both lie on the line

$$(M\gamma+N\beta)\alpha_1+(N\alpha+L\gamma)\beta_1+(L\beta+M\alpha)\gamma_1=0$$

or $$(M\gamma_1+N\beta_1)\alpha+(N\alpha_1+L\gamma_1)\beta+(L\beta_1+M\alpha_1)\gamma=0, \quad (289)$$

and (289) is, therefore, the equation of the polar.

It may be shewn, in a similar manner, that the equations of the polars of the given point with respect to (O_3), (P) and (R) are identical with (273), (274) and (275), respectively.

Again, if we take the non-homogeneous equation

$$(S). \qquad \beta\gamma - k\alpha = 0,$$

we have, by (287),

$$-k(\alpha-\alpha_1) + \gamma_1(\beta-\beta_1) + \beta_1(\nu-\gamma_1) + 2\beta_1\gamma_1 - 2k\alpha_1 = 0;$$

or,
$$k(\alpha + \alpha_1) - \gamma_1\beta - \beta_1\gamma = 0; \tag{290}$$

the same as (278) the equation of the tangent at $(\alpha_1, \beta_1, \gamma_1)$.

Second method.—If $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ be the points of contact of tangents through the given point, we have (Art. 210. (278).), since $(\alpha_1, \beta_1, \gamma_1)$ lies on each tangent,

$$k(\alpha_2 + \alpha_1) - \gamma_2\beta_1 - \beta_2\gamma_1 = 0,$$

and
$$k(\alpha_3 + \alpha_1) - \gamma_3\beta_1 - \beta_3\gamma_1 = 0.$$

Hence,
$$k(\alpha + \alpha_1) - \gamma_1\beta - \beta_1\gamma = 0$$

is the equation of a straight line on which both points of contact lie, and therefore represents the polar of $(\alpha_1, \beta_1, \gamma_1)$.

In the same way the equations of the polars of the point with respect to (T) and (U) may be shewn to be (283) and (285) respectively.

212. *Condition that the straight line (l, m, n) should touch the conic.*

The required condition may be obtained either, as in Art. 131, by comparing the equation of the given line with that of the tangent (Art. 210), or by the direct application of the results of Art. 131 (Eqq. 155, 157); or, again, by combining the equations of the straight line and curve (the latter in the homogeneous form,) and expressing the condition that their points of intersection should be coincident. The following are examples.

$$(N). \qquad L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0.$$

Eliminating α between (N) and the equation

$$l\alpha + m\beta + n\gamma = 0, \tag{291}$$

we get

$$Nm\beta^2 - (Ll - Mm - Nn)\beta\gamma + Mn\gamma^2 = 0;$$

which will give coincident values for $\frac{\beta}{\gamma}$, if

$$(Ll - Mm - Nn)^2 - 4MNmn = 0.$$

This, therefore, is the required condition; it may be written in either of the forms,

$$L^2l^2 + M^2m^2 + N^2n^2 - 2MNmn - 2NLnl - 2LMlm = 0, \tag{292}$$

or

$$(Ll)^{\frac{1}{2}} + (Mm)^{\frac{1}{2}} + (Nn)^{\frac{1}{2}} = 0. \tag{293}$$

Second method.—Identifying (291) with the equation of the tangent at $(\alpha_1, \beta_1, \gamma_1)$ (Art. 210. (270).), we get

$$\frac{M\gamma_1 + N\beta_1}{l} = \frac{N\alpha_1 + L\gamma_1}{m} = \frac{L\beta_1 + M\alpha_1}{n} = -\lambda \text{ (say)}:$$

therefore,

$$\begin{aligned} N\beta_1 + M\gamma_1 + l\lambda &= 0, \\ N\alpha_1 \qquad + L\gamma_1 + m\lambda &= 0, \\ M\alpha_1 + L\beta_1 \qquad + n\lambda &= 0. \end{aligned}$$

Also,

$$l\alpha_1 + m\beta_1 + n\gamma_1 \qquad = 0;$$

and from these, by the elimination of $\alpha_1, \beta_1, \gamma_1$, and λ, we get for the condition of tangency,

$$\begin{vmatrix} 0, & N, & M, & l, \\ N, & 0, & L, & m \\ M, & L, & 0, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0. \tag{294}$$

If the determinant be expanded and the sign of the whole changed, it may be seen that this result agrees with that before obtained (Eqq. (292), (293).), and might have been written down at once as ${}^{l}\underset{n}{\Delta}{}^{m} = 0$, or $\phi(l, m, n)' = 0$ (Art. 131. (155), (157).).

(O_5). $$L^{\frac{1}{2}}\alpha^{\frac{1}{2}} + M^{\frac{1}{2}}\beta^{\frac{1}{2}} + N^{\frac{1}{2}}\gamma^{\frac{1}{2}} = 0.$$

A comparison of the equation of the tangent (Art. 210. (271).) with (291) gives

$$\frac{\alpha_1^{\frac{1}{2}}}{\frac{L^{\frac{1}{2}}}{l}} = \frac{\beta_1^{\frac{1}{2}}}{\frac{M^{\frac{1}{2}}}{m}} = \frac{\gamma_1^{\frac{1}{2}}}{\frac{N^{\frac{1}{2}}}{n}};$$

and substituting these values for $\alpha_1, \beta_1, \gamma_1$ in the equation

$$l\alpha_1 + m\beta_1 + n\gamma_1 = 0,$$

we get for the required condition,

$$\frac{L}{l} + \frac{M}{m} + \frac{N}{n} = 0. \tag{295}$$

(P). $$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 = 0.$$

The condition of tangency may be at once written down from (155) Art. 131 in the form

$$\begin{vmatrix} L^2, & 0, & 0, & l \\ 0, & M^2, & 0, & m \\ 0, & 0, & N^2, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0; \tag{296}$$

or obtained, from (274), as in the last case, in the form

$$\frac{l^2}{L^2}+\frac{m^2}{M^2}+\frac{n^2}{N^2}=0. \tag{297}$$

(*R*). $\beta\gamma - k\alpha^2 = 0.$

Identifying (291) with the equation of the tangent Art. 210. (275), we get

$$\frac{-2k\alpha_1}{l}=\frac{\beta_1}{m}=\frac{\gamma_1}{n};$$

whence, since $\beta_1\gamma_1 = k\alpha_1^2$, we have for the condition sought

$$l^2 - 4kmn = 0. \tag{298}$$

(*S*). $\beta\gamma - k\alpha = 0.$

The equation of the tangent to (*S*) at the point $(\alpha_1, \beta_1, \gamma_1)$, written in the homogeneous form (Art. 210. (278).) is

$$(a\alpha_1 + 2S)\alpha + \left(b\alpha_1 - \frac{2S}{k}\gamma_1\right)\beta + \left(c\alpha_1 - \frac{2S}{k}\beta_1\right)\gamma = 0. \tag{299}$$

Replacing $2S$ in the coefficient of α by $a\alpha_1 + b\beta_1 + c\gamma_1$ (Art. 3. (1).), and comparing (299) with (291), we have

$$\frac{2a\alpha_1 + b\beta_1 + c\gamma_1}{l} = \frac{b\alpha_1 - \frac{2S}{k}\gamma_1}{m} = \frac{c\alpha_1 - \frac{2S}{k}\beta_1}{n} = -\lambda \text{ (suppose).}$$

Whence, eliminating $\alpha_1, \beta_1, \gamma_1, \lambda$, we get, as in (*N*) of this Article,

$$\begin{vmatrix} 2a, & b, & c, & l \\ b, & 0, & -\frac{2S}{k}, & m \\ c, & -\frac{2S}{k}, & 0, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0. \tag{300}$$

If we make use of (157), and remember that, in this case,

$$\phi(\alpha,\beta,\gamma) = a\alpha^2 + 0\beta^2 + 0\gamma^2 - 2\frac{S}{k}\beta\gamma + 2\frac{c}{2}\gamma\alpha + 2\frac{b}{2}\alpha\beta = 0,$$

we shall have

$$\phi(l,m,n)' = -\frac{S^2}{k^2}l^2 - \frac{c^2}{4}m^2 - \frac{b^2}{4}n^2 + 2\left(\frac{bc}{4} + \frac{Sa}{k}\right)mn - 2\left(\frac{Sb}{2k}\right)nl - 2\left(\frac{Sc}{2k}\right)lm, \tag{301}$$

so that the condition is

$$\frac{4S^2l^2}{k^2} + (cm - bn)^2 - \frac{4Slm}{k}\left(\frac{2a}{l} - \frac{b}{m} - \frac{c}{n}\right) = 0; \tag{302}$$

a result which will be found to agree with (300).

(*T*). $\beta\gamma - k^2 = 0.$

Here

$$\phi(\alpha,\beta,\gamma) = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 + 2\left(bc - \frac{2S^2}{k^2}\right)\beta\gamma + 2ca\gamma\alpha + 2ab\alpha\beta,$$

and the condition of tangency (157) becomes

$$\left(\frac{S^2}{k^2} + bc\right)l^2 + almn\left(\frac{a}{l} - \frac{b}{m} - \frac{c}{n}\right) = 0. \tag{303}$$

(*U*). $\beta^2 - k\alpha = 0.$

In this case

$$\phi(l,m,n)' = -\frac{c^2}{4}m^2 - \left(\frac{2Sa}{k} + \frac{b^2}{4}\right)n^2 + 2\left(\frac{bc}{4}\right)mn - 2\left(\frac{Sc}{k}\right)nl, \tag{304}$$

and the condition is that this quantity should equal zero.

213. *To find the co-ordinates of the centre.*

Let, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ be the co-ordinates of the centre. This point (Art. 139) is the pole of the straight line at infinity whose equation is

$$a\alpha + b\beta + c\gamma = 0. \tag{305}$$

(*N*).
$$L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0.$$

The equation of the polar of $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ with respect to this conic (Art. 211. (289).) is

$$(M\bar{\gamma} + N\bar{\beta})\alpha + (N\bar{\alpha} + L\bar{\gamma})\beta + (L\bar{\beta} + M\bar{\alpha})\gamma = 0.$$

Identifying this equation with (305), we get

$$\frac{M\bar{\gamma} + N\bar{\beta}}{a} = \frac{N\bar{\alpha} + L\bar{\gamma}}{b} = \frac{L\bar{\beta} + M\bar{\alpha}}{c} = -\lambda \text{ (say)};$$

whence, proceeding as in Art. 137, we find

$$\frac{\bar{\alpha}}{\begin{vmatrix} N, M, a \\ 0, L, b \\ L, 0, c \end{vmatrix}} = \frac{\bar{\beta}}{-\begin{vmatrix} 0, M, a \\ N, L, b \\ M, 0, c \end{vmatrix}} = \frac{\bar{\gamma}}{\begin{vmatrix} 0, N, a \\ N, L, b \\ M, L, c \end{vmatrix}} = \frac{\lambda}{-\begin{vmatrix} 0, N, M \\ N, 0, L \\ M, L, 0 \end{vmatrix}}; \tag{306}$$

that is,

$$\frac{\frac{\bar{\alpha}}{L}}{-La + Mb + Nc} = \frac{\frac{\bar{\beta}}{M}}{La - Mb + Nc} = \frac{\frac{\bar{\gamma}}{N}}{La + Mb - Nc}$$

$$= \frac{-2S}{L^2a^2 + M^2b^2 + N^2c^2 - 2MNbc - 2NLca - 2LMab} \tag{307}$$

(Prelim. chap (*A*).),

equations which completely determine the centre.

(O_5). $L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 - 2MN\beta\gamma - 2NL\gamma\alpha - 2LM\alpha\beta = 0.$

Comparing the equation of the polar (Art. 210. (273).) with (305), we get

$$\frac{L(-L\bar{\alpha} + M\bar{\beta} + N\bar{\gamma})}{a} = \frac{M(L\bar{\alpha} - M\bar{\beta} + N\bar{\gamma})}{b} = \frac{N(L\bar{\alpha} + M\bar{\beta} - N\bar{\gamma})}{c};$$

whence, proceeding as in the last example, we have finally

$$\frac{\bar{\alpha}}{Mc + Nb} = \frac{\bar{\beta}}{Na + Lc} = \frac{\gamma}{Lb + Ma} = \frac{S}{Lbc + Mca + Nab}. \tag{308}$$

(P). $$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 = 0$$

gives, in a similar way,

$$\frac{\bar{\alpha}}{\frac{a}{L^2}} = \frac{\bar{\beta}}{\frac{b}{M^2}} = \frac{\bar{\gamma}}{\frac{c}{N^2}} = \frac{2S}{\frac{a^2}{L^2} + \frac{b^2}{M^2} + \frac{c^2}{N^2}}; \tag{309}$$

(R). and $$\beta\gamma - k\alpha^2 = 0,$$

$$\frac{\bar{\alpha}}{\frac{a}{2k}} = \frac{\bar{\beta}}{-b} = \frac{\bar{\gamma}}{-c} = \frac{S}{\frac{a^2}{4k} - bc}. \tag{310}$$

(S). $$\beta\gamma - k\alpha = 0.$$

Here, comparing

$$(2a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma})\alpha + \left(b\bar{\alpha} - \frac{2S}{k}\bar{\gamma}\right)\beta + \left(c\bar{\alpha} - \frac{2S}{k}\bar{\beta}\right)\gamma = 0,$$

the *homogeneous* equation of the polar, with (305), we find

$$\frac{\bar{\alpha}}{2bc + \frac{2Sa}{k}} = \frac{\bar{\beta}}{-ca} = \frac{\bar{\gamma}}{-ab} = \frac{k}{a^2}. \tag{311}$$

Similarly, in the two next examples, the subjoined results are obtained.

(*T*).
$$\beta\gamma - k^2 = 0.$$

$$\frac{\bar{\alpha}}{\frac{2S}{a}} = \frac{\bar{\beta}}{0} = \frac{\bar{\gamma}}{0} = 1. \qquad (312)$$

(*U*).
$$\beta^2 - k\alpha = 0.$$

$$\frac{\bar{\alpha}}{c} = \frac{\bar{\beta}}{0} = \frac{\bar{\gamma}}{-a} = \frac{2S}{0}. \qquad (313)$$

214. *Condition that the conic should be a parabola.*

Since (Art. 131. Cor.) the straight line at infinity is a tangent to every parabola; and, again, (Art. 139) the centre of every parabola lies on this line; the required conditions may be deduced from the results of either Art. 212 or Art. 213. They are seen below and will be found to agree with those given in Arts. 185—206.

(*N*). $L^2a^2 + M^2b^2 + N^2c^2 - 2MNbc - 2NLca - 2LMab = 0.$

(*O*$_5$).
$$\frac{L}{a} + \frac{M}{b} + \frac{N}{c} = 0.$$

(*P*).
$$\frac{a^2}{L^2} + \frac{b^2}{M^2} + \frac{c^2}{N^2} = 0.$$

(*R*).
$$k = \frac{a^2}{4bc}.$$

(*S*). *Impossible* (*the curve being always hyperbolic*).

(*T*). *Impossible* (*the locus being always a hyperbola*).

(U). *The condition is satisfied for every value of k (and the curve is always a parabola).*

215. It appears also from Art. 213 that the vertex A of the triangle of reference is the centre of the conic (T) (see Art. 205).

216. *Conditions that the conic should be a circle.*

Applying the conditions (194) of Art. 149 we obtain the following results.

(N).
$$\frac{L}{\alpha}+\frac{M}{\beta}+\frac{N}{\gamma}=0.$$

The conditions are

$$\frac{M}{\sin C}-\frac{N}{\sin B}=\frac{N}{\sin A}-\frac{L}{\sin C}=\frac{L}{\sin B}-\frac{M}{\sin A};$$

whence we easily deduce

$$\frac{L}{\sin A}=\frac{M}{\sin B}=\frac{N}{\sin C}. \tag{314}$$

Hence

$$\frac{\sin A}{\alpha}+\frac{\sin B}{\beta}+\frac{\sin C}{\gamma}=0, \tag{315}$$

or,
$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=0. \tag{316}$$

is the equation of the circle described about the triangle of reference.

(O_5). $L^2\alpha^2+M^2\beta^2+N^2\gamma^2-2MN\beta\gamma-2NL\gamma\alpha-2LM\alpha\beta=0.$

In this case the condition becomes

$$M\sin C+N\sin B=N\sin A+L\sin C=L\sin B+M\sin A;$$

whence we shall get

$$\frac{L}{\cos^2\frac{A}{2}} = \frac{M}{\cos^2\frac{B}{2}} = \frac{N}{\cos^2\frac{C}{2}}; \qquad (317)$$

and *the equation of the inscribed circle will be*

$$\cos\frac{A}{2}\alpha^{\frac{1}{2}} + \cos\frac{B}{2}\beta^{\frac{1}{2}} + \cos\frac{C}{2}\gamma^{\frac{1}{2}} = 0. \qquad (318)$$

(P). $$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 = 0.$$

The conditions of Art. 149 become, in this case,

$$M^2\sin^2 C + N^2\sin^2 B = N^2\sin^2 A + L^2\sin^2 C = L^2\sin^2 B + M^2\sin^2 A,$$

and give

$$\frac{L^2}{\sin 2A} = \frac{M^2}{\sin 2B} = \frac{N^2}{\sin 2C}, \qquad (319)$$

and *the equation of the self-conjugate circle is*

$$\sin 2A\,\alpha^2 + \sin 2B\,\beta^2 + \sin 2C\,\gamma^2 = 0; \qquad (320)$$

or, $$a\cos A\,\alpha^2 + b\cos B\,\beta^2 + c\cos C\,\gamma^2 = 0; \qquad (321)$$

which will not, however, represent a real locus (Art. 194) unless one of the coefficients be negative, that is, *unless the triangle of reference be obtuse-angled.*

(R). $$\beta\gamma - k\alpha^2 = 0.$$

Here, the conditions (Art. 149) are

$$\sin B\sin C = k\sin^2 C = k\sin^2 B;$$

whence, $k = 1$, *and the triangle of reference is isosceles.* (322)

EQUATION OF A CONIC REFERRED TO TWO TANGENTS AND THEIR CHORD OF CONTACT.

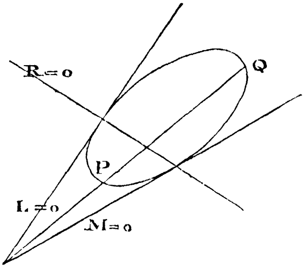
217. Let the equation of a conic referred to a pair of tangents and their chord of contact (Art. 174. (*G*).) be

$$LM = R^2. \qquad (323)$$

Then, if

$$\frac{L}{1} = \frac{M}{\mu^2} \qquad (324)$$

Fig. 37.



be the equation of any chord PQ, through the intersection of the fixed tangents, we get, by combining (323) and (324), the equations

$$\frac{L}{1} = \frac{R}{\pm\mu} = \frac{M}{\mu^2}, \qquad (325)$$

to determine its points of intersection with the conic; the upper and lower signs referring respectively to the points P and Q, which lie on opposite sides of $R = 0$.

218. The chord PO may be denoted briefly by μ^2, and the points P and Q (after Dr. Salmon's notation) by $+\mu$ and $-\mu$ respectively.

219. The reader will observe that the equations

$$R = \pm\mu L,$$

$$M = \pm\mu R,$$

represent the pairs of lines which join (LR) and (MR) respectively to P and Q.

Similarly, any pair of straight lines through (LR) and (MR) whose equations are of the form

$$R = kL,$$
$$M = kR,$$

will intersect on the curve.

220. Any point which does not lie on the curve may be denoted by a pair of equations of the form

$$(R = hL,\ M = kR).$$

221. *To find the equation of the chord which joins two given points on the conic* $LM = R^2$.

Let μ_1, μ_2 (Art. 218) be the two points. At these points, as in Art. 217. (325), we have, respectively,

$$\frac{L}{1} = \frac{R}{\mu_1} = \frac{M}{\mu_1^2} \qquad (326)$$

and

$$\frac{L}{1} = \frac{R}{\mu_2} = \frac{M}{\mu_2^2}. \qquad (327)$$

Suppose the equation of the chord to be

$$lL + rR + mM = 0. \qquad (328)$$

Since μ_1, μ_2 both lie on this line, we have, by (326) and (327),

$$l + r\mu_1 + m\mu_1^2 = 0, \qquad (329)$$
$$l + r\mu_2 + m\mu_2^2 = 0. \qquad (330)$$

Hence, eliminating l, m and r between (328), (329) and (330), we get for the equation of the chord $\mu_1\mu_2$

$$\begin{vmatrix} L, & R, & M \\ 1, & \mu_1, & \mu_1^2 \\ 1, & \mu_2, & \mu_2^2 \end{vmatrix} = 0\,; \qquad (331)$$

which, when expanded and divided through by $\mu_1 - \mu_2$, becomes

$$\mu_1\mu_2 L - (\mu_1 + \mu_2)R + M = 0. \qquad (332)$$

222. *To find the equation of the tangent at any point μ_1 on the conic $LM = R^2$.*

Making $\mu_2 = \mu_1$, in the result of the last Article, we see that the equation of the tangent at μ_1 is

$$\mu_1^2 L - 2\mu_1 R + M = 0. \tag{333}$$

223. Similarly the equation of the tangent at $-\mu_1$ (Art. 218), the other extremity of the chord μ_1^2, is

$$\mu_1^2 L + 2\mu_1 R + M = 0. \tag{334}$$

224. *To find the polar of a given point with respect to the conic $LM = R^2$.*

Suppose the given point (Art. 220) to be the intersection of the pair of lines

$$(R = hL,\ R = kM), \tag{335}$$

and let μ_1 be the point of contact of one of the tangents through this point. The equation of the tangent (Art. 222. (333).) is

$$\mu_1^2 L - 2\mu_1 R + M = 0.$$

Since therefore the point (335) lies on this line, we have, by substitution

$$k\mu_1^2 - 2hk\mu_1 + h = 0. \tag{336}$$

But at the point of contact μ_1 we have (Art. 221. (326).)

$$\frac{L}{1} = \frac{R}{\mu_1} = \frac{M}{\mu_1^2}; \tag{337}$$

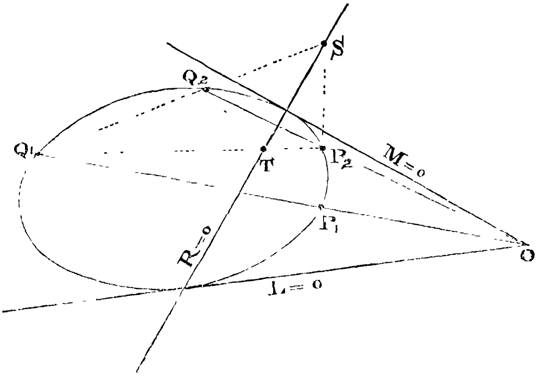
and, eliminating μ_1 between (336) and (337), we find for the equation of the polar at the given point

$$hL - 2hkR + kM = 0. \tag{338}$$

225. *If from a given point O two straight lines be drawn cutting a conic in the points P_1 and Q_1, P_2 and Q_2, respectively; the two pairs of chords which join, directly and transversely, these four points, will intersect on the polar of O.*

Let $L = 0$, $M = 0$ be the equations of the pair of tangents which can be drawn to the conic through the given point: and let $R = 0$ represent the polar of O. The equation of the conic referred to these three lines will be

Fig. 38.



$$LM - R^2 = 0\,; \tag{339}$$

and, if the straight lines OP_1Q_1, OP_2Q_2 be represented by μ_1^2, μ_2^2, respectively (Art. 218), the points P_1, Q_1 will be denoted by $\pm\mu_1$, and P_2, Q_2 by $\pm\mu_2$, respectively. Hence (Art 221. (332).) the chords P_1P_2, Q_1Q_2 will be represented by

$$\mu_1\mu_2 L \pm (\mu_1 + \mu_2)R + M = 0, \tag{340}$$

and the transverse pair, P_1Q_2, P_2Q_1, by

$$-\mu_1\mu_2 L \pm (\mu_1 - \mu_2)R + M = 0. \tag{341}$$

From the form of these equations it follows that the two pairs have their points of intersection on $R=0$, the polar of O. For (340) shews (Art. 16) that P_1Q_1, P_2Q_2, both pass through S, the point of intersection of $R=0$ with $M+\mu_1\mu_2L=0$; and from (341) it appears that P_1Q_2, P_2Q_1 both pass through T, the point of intersection of $R=0$ and $M-\mu_1\mu_2L=0$.

226. Hence also it follows that the equations of OS, OT are respectively

$$M+\mu_1\mu_2L=0, \tag{342}$$

$$M-\mu_1\mu_2L=0. \tag{343}$$

227. Therefore also OS, OT form with OP_1, OP_2 a harmonic pencil (Art. 95); for the equations of these four lines are

$$L=0,\quad M=0,\quad M\pm\mu_1\mu_2L=0.$$

228. Again, from the form of the equations (340) it appears that the four lines SO, ST, SP_1, SQ_1 form a harmonic pencil. Therefore the chords P_1Q_1, P_2Q_2 are cut harmonically by the point O and its polar; and we have the well-known Theorem;—

Every chord of a conic is harmonically divided by any point on it and the polar of that point.

This is also evident from the form of the equations

$$R=\pm\mu_1L, \tag{344}$$

$$R=\pm\mu_2L, \tag{345}$$

representing (Art. 219) the pairs of lines which join the point of contact of one of the fixed tangents to the extremities of the chords P_1Q_1, P_2Q_2 respectively. For (344) and (345) each form a harmonic pencil with $L=0$, $R=0$, and, consequently, the transversals OP_1Q_1, OP_2Q_2 are harmonically divided.

229. *The poles, with reference to a given conic, of straight lines which pass through a fixed point, lie on a fixed right line.*

Let the conic be referred to any pair of tangents and their chord of contact, and let its equation be

$$LM = R^2. \tag{346}$$

Also let the straight line the locus of whose pole with respect to (346) is to be found be represented by the equation

$$lL - 2\rho R + mM = 0, \tag{347}$$

in which ρ is indeterminate (Art. 16).

Suppose

$$(R = hL,\ R = kM) \tag{348}$$

(Art. 224. (335).), to be the pole of (347); then (Art. 224. (338).) the equation

$$hL - 2hkR + kM = 0 \tag{349}$$

must be identical with (347). Comparing them, we get

$$\frac{h}{l} = \frac{hk}{\rho} = \frac{k}{m}. \tag{350}$$

But at the pole we have, by (348),

$$\frac{L}{k} = \frac{R}{hk} = \frac{M}{h}; \tag{351}$$

hence, substituting for h, k, from (350) in (351), we get

$$\frac{L}{m} = \frac{R}{\rho} = \frac{M}{l};$$

and the locus of the pole of (348) is a straight line whose equation is

$$lL - mM = 0. \tag{352}$$

230. We shall conclude this Chapter by calling the attention of the reader to a few geometrical theorems, which are involved in the forms of equations investigated in the preceding Articles.

Thus from the forms of (R), (S), (T) and (U) respectively we deduce the following;—

The product of the distances of any point on a conic from a pair of tangents bears a constant ratio to the square of its distance from their chord of contact.

If at the extremities of any chord of a hyperbola parallels be drawn to the asymptotes, the distance of any point on the curve from this chord is in a constant ratio to the product of its distances from these two straight lines.

The rectangle under the distances of any point on a hyperbola from the asymptotes is constant.

The square of the distance of any point on a parabola from a diameter is proportional to its distance from the tangent at its extremity.

Again, (316) may be written in the form

$$a\beta\gamma + b\gamma a + ca\beta = 0\,;$$

and, if P be any point on the circumscribed circle and Q, R, S the feet of the perpendiculars from P upon the sides of the triangle, this equation asserts that the algebraical sum of the triangular areas PRS, PSQ and PQR is $= 0$. Hence Q, R, S are in the same straight line; and we have the following theorem;—

If from a point on the circumference of a circle perpendiculars be drawn to the sides of any inscribed triangle, the feet of these perpendiculars will lie in one and the same straight line.

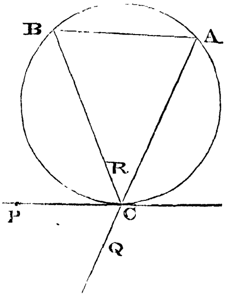
Also from (322) it appears that *if a circle be described touching the sides of an isosceles triangle at the extremities of the base, the rectangle under the distances of any point on the circumference from the two sides is equal to the square of its distance from the base.*

CHAPTER IX.

THE CIRCLE.

231. *To find the equation of the circle described about the triangle ABC.*

Fig. 39.



We may assume the equation (Art. 180. (*N*).) to be of the form

$$L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0. \tag{353}$$

The equation of the tangent at C (Art. 181. (234).) gives for any point P upon it

$$\frac{L}{M} = -\frac{\alpha}{\beta}.$$

Therefore (Euc. iii. 32), when (353) represents a circle,

$$\frac{L}{M} = -\frac{\alpha}{\beta} = \frac{PR}{PQ} = \frac{PC\sin A}{PC\sin B} = \frac{\sin A}{\sin B};$$

and we have, by symmetry,

$$\frac{L}{\sin A} = \frac{M}{\sin B} = \frac{N}{\sin C}.$$

Hence (353) becomes

$$\sin A\,\beta\gamma + \sin B\,\gamma\alpha + \sin C\,\alpha\beta = 0, \tag{354}$$

or,

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0. \tag{355}$$

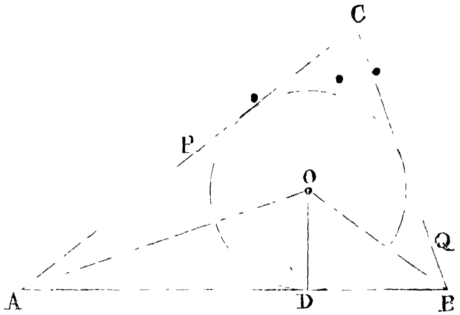
232. *To find the equations of the inscribed and escribed circles.*

The equation must (Art. 187. (O_5).) be of the form

$$(L\alpha)^{\frac{1}{2}} + (M\beta)^{\frac{1}{2}} + (N\gamma)^{\frac{1}{2}} = 0\,; \qquad (356)$$

and the equation of the straight line which joins C to the point of contact of the opposite side (Art. 187. (238).) gives, for *any* point upon it and therefore for the point of contact D,

Fig. 40.



$$\frac{L}{M} = \frac{\beta}{\alpha} = \frac{DR}{DQ} = \frac{DA\sin A}{DB\sin B} = \frac{OD\cot\frac{A}{2}\sin A}{OD\cot\frac{B}{2}\sin B} = \frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}.$$

Hence, by symmetry,

$$\frac{L}{\cos^2\frac{A}{2}} = \frac{M}{\cos^2\frac{B}{2}} = \frac{N}{\cos^2\frac{C}{2}},$$

and the required equation is, by (356),

$$\cos\frac{A}{2}\alpha^{\frac{1}{2}} + \cos\frac{B}{2}\beta^{\frac{1}{2}} + \cos\frac{C}{2}\gamma^{\frac{1}{2}} = 0. \qquad (357),$$

233. Similarly the equations of the escribed circles may be shewn to be

$$\cos\frac{A}{2}(-\alpha)^{\frac{1}{2}} + \sin\frac{B}{2}\beta^{\frac{1}{2}} + \sin\frac{C}{2}\gamma^{\frac{1}{2}} = 0,$$

$$\sin\frac{A}{2}\,\alpha^{\frac{1}{2}} + \cos\frac{B}{2}\,(-\beta)^{\frac{1}{2}} + \sin\frac{C}{2}\,\gamma^{\frac{1}{2}} = 0, \qquad (358)$$

$$\sin\frac{A}{2}\,\alpha^{\frac{1}{2}} + \sin\frac{B}{2}\,\beta^{\frac{1}{2}} + \cos\frac{C}{2}\,(-\gamma)^{\frac{1}{2}} = 0.$$

234. It will be seen that the equations just obtained for the circumscribed and inscribed circles are the same as those given in Art. 216. A full discussion of them here would be superfluous, since all that was said with regard to the conics (N) and (O_5) in the two preceding Chapters applies, *mutatis mutandis*, to these circles. A few, however, of the more important results there obtained will be given in the present Chapter under their modified forms. For the sake of clearness we commence by explaining the notation employed.

235.

R, O_R;	A, B, C	represent respectively the *radius, centre,* and *points of intersection* (or *contact*) *with the sides of the triangle of reference, of*	$S_R = 0$,	*the circumscribed* ⊙.
r, O_r;	A_r, B_r, C_r		$S_r = 0$,	. . *inscribed* . .
r_a, O_a;	A_a, B_a, C_a		$S_a = 0$,	. . *escribed* (*on* BC) . .
r_b, O_b;	A_b, B_b, C_b		$S_b = 0$,	. . *escribed* (*on* CA) . .
r_c, O_c;	A_c, B_c, C_c		$S_c = 0$,	. . *escribed* (*on* AB) . .
r_s, O_s;			$S_s = 0$,	. . *self-conjugate* . .
r_9, O_9;	A_1, B_1, C_1 A_2, B_2, C_2		$S_9 = 0$,	. . *nine-point* . .

Similarly S_{abc}, S_{rbc}, S_{arc}, S_{abr} are the circles described through O_a, O_b, O_c; O_r, O_b, O_c; O_a, O_r, O_c; O_a, O_b, O_r, respectively.

Also,	A_1, B_1, C_1	are the	*middle points of the sides.*
	A_2, B_2, C_2		*feet of* $\perp^{rs}$ *from the vertices;*

and

P_1	is the intersection of	the $\perp$'s at the middle points.
P_2		the $\perp$rs from the vertices.
G_1		AA_1, BB_1, CC_1.
G_r		AA_r, BB_r, CC_r.
G_a		AA_a, BB_a, CC_c.
G_b		AA_b, BB_b, CC_b.
G_c		AA_c, BB_c, CC_c.
G_{abc}		AA_a, BB_b, CC_c.

236.

(S_R). $$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0.$$

Suppose a triangle $A'B'C'$ to be formed by drawing tangents to the circumscribed circle at the points A, B, C. Then it will appear, as in Art. 181, that its sides, whose equations are

$$\frac{\beta}{b} + \frac{\gamma}{c} = 0,$$

$$\frac{\gamma}{c} + \frac{\alpha}{a} = 0, \tag{359}$$

and $$\frac{\alpha}{a} + \frac{\beta}{b} = 0,$$

meet the opposite sides of the original triangle in points which lie on the straight line

$$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0; \tag{360}$$

also Art. 182. (236).) AA', BB', CC' are represented by the equations

$$\frac{\beta}{b} - \frac{\gamma}{c} = 0,$$

$$\frac{\gamma}{c} - \frac{\alpha}{a} = 0, \qquad (361)$$

and
$$\frac{\alpha}{a} - \frac{\beta}{b} = 0,$$

and (Art. 183. (237).) meet in a point $(\alpha_0, \beta_0, \gamma_0)$ such that

$$\frac{\alpha_0}{a} = \frac{\beta_0}{b} = \frac{\gamma_0}{c} = \frac{2S}{a^2 + b^2 + c^2}. \qquad (362)$$

237. For the *equation of the chord* joining the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, we shall have (Art. 209. (249).)

$$\frac{a\alpha}{\alpha_1\alpha_2} + \frac{b\beta}{\beta_1\beta_2} + \frac{c\gamma}{\gamma_1\gamma_2} = 0, \qquad (363)$$

and the *equation of the tangent at* $(\alpha_1, \beta_1, \gamma_1)$ will be

$$\frac{a\alpha}{\alpha_1^2} + \frac{b\beta}{\beta_1^2} + \frac{c\gamma}{\gamma_1^2} = 0; \qquad (364)$$

while (Art. 212. (293).), for the *condition of tangency of* (l, m, n), we have

$$(al)^{\frac{1}{2}} + (bm)^{\frac{1}{2}} + (cn)^{\frac{1}{2}} = 0. \qquad (365)$$

238. *The centre* O_R *of the circumscribed circle is given* (Art. 213. (307).) *by the equations*

$$\frac{\bar{\alpha}}{\cos A} = \frac{\bar{\beta}}{\cos B} = \frac{\bar{\gamma}}{\cos C} = \frac{2S}{a\cos A + b\cos B + c\cos C}$$
$$= \frac{abc}{4S}$$
$$= R, \qquad (366)$$

and therefore, as may easily be shewn, coincides with P_1.

239.

$$(S_{,}). \qquad \cos\frac{A}{2}\alpha^{\frac{1}{2}} + \cos\frac{B}{2}\beta^{\frac{1}{2}} + \cos\frac{C}{2}\gamma^{\frac{1}{2}} = 0.$$

It may be shewn, as in Art. 187, that the equations of $AA_{,}$, $BB_{,}$, $CC_{,}$ are

$$\begin{aligned} \cos^2\frac{B}{2}\beta - \cos^2\frac{C}{2}\gamma &= 0, \\ \cos^2\frac{C}{2}\gamma - \cos^2\frac{A}{2}\alpha &= 0, \qquad (367) \\ \cos^2\frac{A}{2}\alpha - \cos^2\frac{B}{2}\beta &= 0, \end{aligned}$$

respectively, and that $G_{,}$, their point of intersection, is given by the equations

$$\cos^2\frac{A}{2}\alpha_0 = \cos^2\frac{B}{2}\beta_0 = \cos^2\frac{C}{2}\gamma_0, \qquad (368)$$

or, $$a(s-a)\alpha_0 = b(s-b)\beta_0 = c(s-c)\gamma_0. \qquad (369)$$

240. The *equation of the tangent* at any point $(\alpha_1, \beta_1, \gamma_1)$ (Art. 210. (271).) is

$$\frac{1}{\alpha_1^{\frac{1}{2}}}\cos\frac{A}{2}\alpha + \frac{1}{\beta_1^{\frac{1}{2}}}\cos\frac{B}{2}\beta + \frac{1}{\gamma_1^{\frac{1}{2}}}\cos\frac{C}{2}\gamma = 0, \qquad (370)$$

and the *condition of tangency* for (l, m, n) (Art. 212. (295).) is

$$\frac{1}{l}\cos^2\frac{A}{2} + \frac{1}{m}\cos^2\frac{B}{2} + \frac{1}{n}\cos^2\frac{C}{2} = 0; \qquad (371)$$

or, $$\frac{a(s-a)}{l} + \frac{b(s-b)}{m} + \frac{c(s-c)}{n} = 0. \qquad (372)$$

241. Again, O_r *the centre of the inscribed circle* (Art. 213. (308).) *is given by the equations*

$$\bar{\alpha} = \bar{\beta} = \bar{\gamma} = \frac{S}{s} = r, \tag{373}$$

and therefore (Art. 20. (10).) coincides, as we know from the geometry to be the case, with the point of intersection of the bisectors of the angles of the triangle

242. *The centre O_a of the circle escribed on BC* (since it is the intersection of the lines $\beta - \gamma = 0$, $\gamma + \alpha = 0$, $\alpha + \beta = 0$) is given by the equations

$$-\bar{\alpha} = \bar{\beta} = \bar{\gamma} = \frac{2S}{b + c - a}$$

$$= \frac{S}{s - a} = r_a. \tag{374}$$

243.

(S_s). $$a \cos A\alpha^2 + b \cos B\beta^2 + c \cos C\gamma^2 = 0.$$

The *equation of the tangent* at any point $(\alpha_1, \beta_1, \gamma_1)$ on the self-conjugate circle (Art. 210. (274).) is

$$a \cos A\alpha_1\alpha + b \cos B\beta_1\beta + c \cos C\gamma_1\gamma = 0, \tag{375}$$

and the *condition of tangency* for (l, m, n) (Art. 212. (297).) is

$$\frac{l^2}{a \cos A} + \frac{m^2}{b \cos B} + \frac{n^2}{c \cos C} = 0. \tag{376}$$

244. Also (Art. 213. (309).) O_s, the centre of the self-conjugate circle, is such that

$$\cos A\bar{\alpha} = \cos B\bar{\beta} = \cos C\bar{\gamma} = \frac{2S}{a \sec A + b \sec B + c \sec C}$$

$$= 2R \cos A \cos B \cos C, \tag{377}$$

and therefore (Art. 25. (18).) coincides with P_2.

THE NINE-POINT CIRCLE.

245. *Theorem.—In any triangle ABC, the feet of the perpendiculars from the vertices (which meet in P_2), the bisections of the segments AP_2, BP_2, CP_2, and the middle points of the sides, are nine points which lie on the same circle; and this circle touches the inscribed and the three escribed circles of the triangle.* (Nouvelles Annales de Mathématiques, 1842).

The following general form of this Theorem is given by M. Terquem in the same paper.

If through the vertices of any triangle ABC, inscribed in a conic, three straight lines be drawn conjugate to the opposite sides[a], *these three lines intersect in a point P; and the three points in which they meet the opposite sides of the triangle, the bisections of the segments AP, BP, CP, and the middle points of the sides, are nine points which lie on a second similar and similarly situated conic: also this conic touches an inscribed conic which is similar to the given one and similarly placed.*

The equation of the nine-point circle will be found hereafter (Arts. 253, 254).

246. *To shew that the equation of a circle may always be written in the form*

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma);$$

l, m, n being arbitrary constants.

The general equation of the second degree

$$\phi(\alpha, \beta, \gamma) = A\alpha^2 + B\beta^2 + C\gamma^2 + 2D\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta = 0$$

may, since $a\alpha + b\beta + c\gamma = 2S$, be written in the form

$$Abc(b\beta + c\gamma - 2S)\alpha + Bca(c\gamma + a\alpha - 2S)\beta + Cab(a\alpha + b\beta - 2S)\gamma$$
$$-2abc(D\beta\gamma + E\gamma\alpha + F\alpha\beta) = 0,$$

---

[a] Straight lines are said to be *conjugates* with respect to a conic when they are parallel to conjugate diameters.

or,

$$a(Bc^2+Cb^2-2Dbc)\beta\gamma+b(Ca^2+Ac^2-2Eca)\gamma\alpha+c(Ab^2+Ba^2-2Fab)\alpha\beta$$

$$-2abcS\left(\frac{A\alpha}{a}+\frac{B\beta}{b}+\frac{C\gamma}{c}\right)=0. \qquad (378)$$

But, if $\phi(\alpha, \beta, \gamma) = 0$ represent a circle we have (Art. 149. (195).)

$$Bc^2+Cb^2-2Dbc = Ca^2+Ac^2-2Eca = Ab^2+Ba^2-2Fab$$

$$= k \text{ (say)}; \qquad (379)$$

and (378) becomes

$$\alpha\beta\gamma+b\gamma\alpha+c\alpha\beta=\frac{2S}{k}\,abc\left(\frac{A}{a}\alpha+\frac{B}{b}\beta+\frac{C}{c}\gamma\right) \qquad (380)$$

$$=\frac{abc}{k}(a\alpha+b\beta+c\gamma)\left(\frac{A}{a}\alpha+\frac{B}{b}\beta+\frac{C}{c}\gamma\right), \qquad (381)$$

which is of the required form, the values of l, m and n being $\frac{Abc}{k}$, $\frac{Bca}{k}$, $\frac{Cab}{k}$, respectively.

For examples of this form of equation to the circle the reader is referred to Arts. 252 et sqq.

247. The form of the equation arrived at in the last Article shews (Art. 169. (B).) that every circle meets the circle

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=0 \qquad (382)$$

in four points, two of which are imaginary, the straight line at infinity (Art. 52) being always *one* of the chords of intersection. Hence *all circles pass through the same two imaginary points at infinity.*

If this property be assumed, it follows at once from Art. 169 that the equation of a circle may be written in the form

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma), \quad (383)$$

or $$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 2S(l\alpha + m\beta + n\gamma). \quad (384)$$

248. The equation of the *other* chord of intersection is evidently

$$l\alpha + m\beta + n\gamma = 0; \quad (385)$$

this therefore is the equation of the radical axis of (383) and the circumscribed circle (382).

249. *To find the equation of the radical axis of two circles.*

Let the equations of the circles, when thrown into the form which has just been investigated, be

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l_1\alpha + m_1\beta + n_1\gamma),$$

and $$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l_2\alpha + m_2\beta + n_2\gamma).$$

Subtracting, we get

$$(a\alpha + b\beta + c\gamma)\{(l_1 - l_2)\alpha + (m_1 - m_2)\beta + (n_1 - n_2)\gamma\} = 0: \quad (386)$$

hence, one of the chords of intersection (Art. 167) is imaginary and at an infinite distance (see also Art. 247); the other is real [except in the case of concentric circles (Art. 251),] and is represented by

$$(l_1 - l_2)\alpha + (m_1 - m_2)\beta + (n_1 - n_2)\gamma = 0, \quad (387)$$

which is therefore the equation of the radical axis.

250. If the equations of the two circles be given in the general form

$$A_1\alpha^2 + B_1\beta^2 + C_1\gamma^2 + 2D_1\beta\gamma + 2E_1\gamma\alpha + 2F_1\alpha\beta = 0,$$

$$A_2\alpha^2 + B_2\beta^2 + C_2\gamma^2 + 2D_2\beta\gamma + 2E_2\gamma\alpha + 2F_2\alpha\beta = 0, \quad (388)$$

the equation of their radical axis (Art. 246. (381).) will be

$$\frac{1}{a}\left(\frac{A_1}{k_1}-\frac{A_2}{k_2}\right)\alpha+\frac{1}{b}\left(\frac{B_1}{k_1}-\frac{B_2}{k_2}\right)\beta+\frac{1}{c}\left(\frac{C_1}{k_1}-\frac{C_2}{k_2}\right)\gamma=0; \quad (389)$$

the values of k_1, k_2 being similar to that which k has in the Article referred to.

251. Suppose the given equations to differ only by a constant term, or, which is the same thing, by a multiple of $a\alpha+b\beta+c\gamma$. The circles, in this case, are concentric[b], and we get on subtraction

$$(a\alpha+b\beta+c\gamma)^2=0;$$

which shews (Art. 249. (386).) that both chords of intersection are imaginary and coincide with the straight line at infinity. Hence it is to be inferred that *concentric circles touch each other in two imaginary points at infinity.*

252. *To find the equation of the circle inscribed in the triangle ABC.*

Let it be assumed (Art. 247. (383).) to be

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=(a\alpha+b\beta+c\gamma)(l\alpha+m\beta+n\gamma). \quad (390)$$

---

[b] For the centre of the conic

$$\phi_1(\alpha,\beta,\gamma)+2k(a\alpha+b\beta+c\gamma)=0$$

is (Art. 137. (166).) given by the equations

$$\frac{\left(\frac{d\phi_1}{d\alpha}\right)+2ka}{a}=\frac{\left(\frac{d\phi_1}{d\beta}\right)+2kb}{b}=\frac{\left(\frac{d\phi_1}{d\gamma}\right)+2kc}{c}.$$

Therefore,

$$\frac{\left(\frac{d\phi_1}{d\alpha}\right)}{a}=\frac{\left(\frac{d\phi_1}{d\beta}\right)}{b}=\frac{\left(\frac{d\phi_1}{d\gamma}\right)}{c},$$

and the conic is concentric with the conic $\phi_1(\alpha,\beta,\gamma)=0$.

Since the circle passes through A_r, $(0, (s-c)\sin C, (s-b)\sin B)$, or (Art. 26), $\left(0, \frac{1}{b(s-b)}, \frac{1}{c(s-c)}\right)$, the point of contact of the side BC, we have by substitution in (390)

$$\frac{a}{bc(s-b)(s-c)} = \left(\frac{1}{s-b} + \frac{1}{s-c}\right)\left(\frac{m}{b(s-b)} + \frac{n}{c(s-c)}\right).$$

Hence, $$\frac{m}{b(s-b)} + \frac{n}{c(s-c)} = \frac{a}{abc}, \tag{391}$$

Similarly, $$\frac{n}{c(s-c)} + \frac{l}{a(s-a)} = \frac{b}{abc}, \tag{392}$$

and $$\frac{l}{a(s-a)} + \frac{m}{b(s-b)} = \frac{c}{abc}. \tag{393}$$

Therefore, subtracting (391) from the sum of (392) and (393), we get

$$\frac{2l}{a(s-a)} = \frac{b+c-a}{abc} = \frac{2(s-a)}{abc}.$$

and therefore, by symmetry,

$$\frac{l}{a(s-a)^2} = \frac{m}{b(s-b)^2} = \frac{n}{c(s-c)^2} = \frac{1}{abc}, \tag{394}$$

and (390) becomes

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = \frac{2S}{abc}\left\{a(s-a)^2\alpha + b(s-b)^2\beta + c(s-c)^2\gamma\right\}, \tag{395}$$

which is the required equation.

253. *To find the equation of the circle which passes through the middle points of the sides of the triangle ABC.*

Suppose the equation to be

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma). \tag{396}$$

Since the circle passes through the middle point A_1

$$\left(0, \frac{a}{2}\sin C, \frac{a}{2}\sin B\right) \text{ or } \left(0, \frac{1}{b}, \frac{1}{c}\right), \text{ (396) gives}$$

$$\frac{a^2}{abc} = 2\left(\frac{m}{b} + \frac{n}{c}\right).$$

Therefore,
$$\frac{m}{b} + \frac{n}{c} = \frac{a^2}{2abc}. \tag{397}$$

Similarly,
$$\frac{n}{c} + \frac{l}{a} = \frac{b^2}{2abc}, \tag{398}$$

and
$$\frac{l}{a} + \frac{m}{b} = \frac{c^2}{2abc}. \tag{399}$$

Subtracting (397) from the sum of (398) and (399), we have

$$\frac{2l}{a} = \frac{b^2 + c^2 - a^2}{2abc} = \frac{\cos A}{a};$$

therefore, by symmetry,

$$\frac{l}{\cos A} = \frac{m}{\cos B} = \frac{n}{\cos C} = \frac{1}{2}, \tag{400}$$

and, substituting these values of l, m, n in (396), we have

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = \frac{1}{2}(a\alpha + b\beta + c\gamma)(\cos A\,\alpha + \cos B\,\beta + \cos C\,\gamma). \tag{401}$$

254. *To find the equation of the circle which passes through the feet of the perpendiculars from the vertices of the triangle ABC upon the opposite sides.*

Assuming the equation to be

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma), \tag{402}$$

we have, since it passes through A_2 (see Art. 235) $(0, p_a \cos C, p_a \cos B)$ or $\left(0, \frac{1}{\cos B}, \frac{1}{\cos C}\right)$,

$$\frac{a}{\cos B \cos C} = \left(\frac{b}{\cos B} + \frac{c}{\cos C}\right)\left(\frac{m}{\cos B} + \frac{n}{\cos C}\right);$$

that is, since $b \cos C + c \cos B = a$,

$$\frac{m}{\cos B} + \frac{n}{\cos C} = 1. \tag{403}$$

Similarly, $$\frac{m}{\cos C} + \frac{l}{\cos A} = 1, \tag{404}$$

and $$\frac{l}{\cos A} + \frac{m}{\cos B} = 1. \tag{405}$$

Hence, subtracting (403) from the sum of (404) and (405), we find

$$\frac{2l}{\cos A} = 1,$$

and therefore, by symmetry,

$$\frac{l}{\cos A} = \frac{m}{\cos B} = \frac{n}{\cos C} = \frac{1}{2}; \tag{406}$$

values which give for the required equation

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = \frac{1}{2}(a\alpha + b\beta + c\gamma)(\cos A\,\alpha + \cos B\,\beta + \cos C\gamma). \tag{407}$$

255. The resulting equations of the last two Articles, it will be observed, are identical, and M. Terquem's Theorem (Art. 245) is proved so far as the six points $A_1, B_1, C_1, A_2, B_2, C_2$ are concerned. With regard to the three remaining points it is only necessary to remark that the circle which passes through A_2, B_2, C_2 must likewise, by what precedes, (since these points are also the feet of the

perpendiculars upon the sides of the triangle ABP_2 from the vertices which subtend them,) pass through the middle points of AP_2, BP_2, and, for a similar reason, must also bisect CP_2.

256. Thus it appears that (407) is the *equation of the nine-point circle* S_9. That part of the Theorem of Art. 245 which refers to the contact of this circle with the inscribed and escribed circles of the triangle ABC may be verified by applying the condition of tangency (Art. 131. (155).) to the equation of the radical axis (see Art. 262) of S_9 with S_r, S_a, S_b and S_c successively.

257. Fig. 41 exhibits the relations in which these circles stand to one another, and the position with respect to them of certain important points connected with the triangle of reference. Thus O_R, G_1, O_9 and P_2 are collinear, O_9 being the bisection of P_2O_R, and $G_1P_2 = 2G_1O_R$. Again O_r, G_1 and G_{abc} are collinear, and $G_1G_{abc} = 2G_1O_r$. Hence also P_2G_{abc} is parallel to O_rO_R, and $P_2G_{abc} = 2O_rO_R$.

258. The *centre of the nine-point circle* is manifestly the intersection of perpendiculars to the segments A_1A_2, B_1B_2, C_1C_2, at their middle points.

If therefore $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ be the co-ordinates of O_9, we shall have (Art. 238. (366).) and (Art. 244. (377).),

$$\bar{\alpha} = \frac{1}{2}(R\cos A + 2R\cos B\cos C) = \frac{R}{2}\cos(B-C),$$

and therefore, by symmetry,

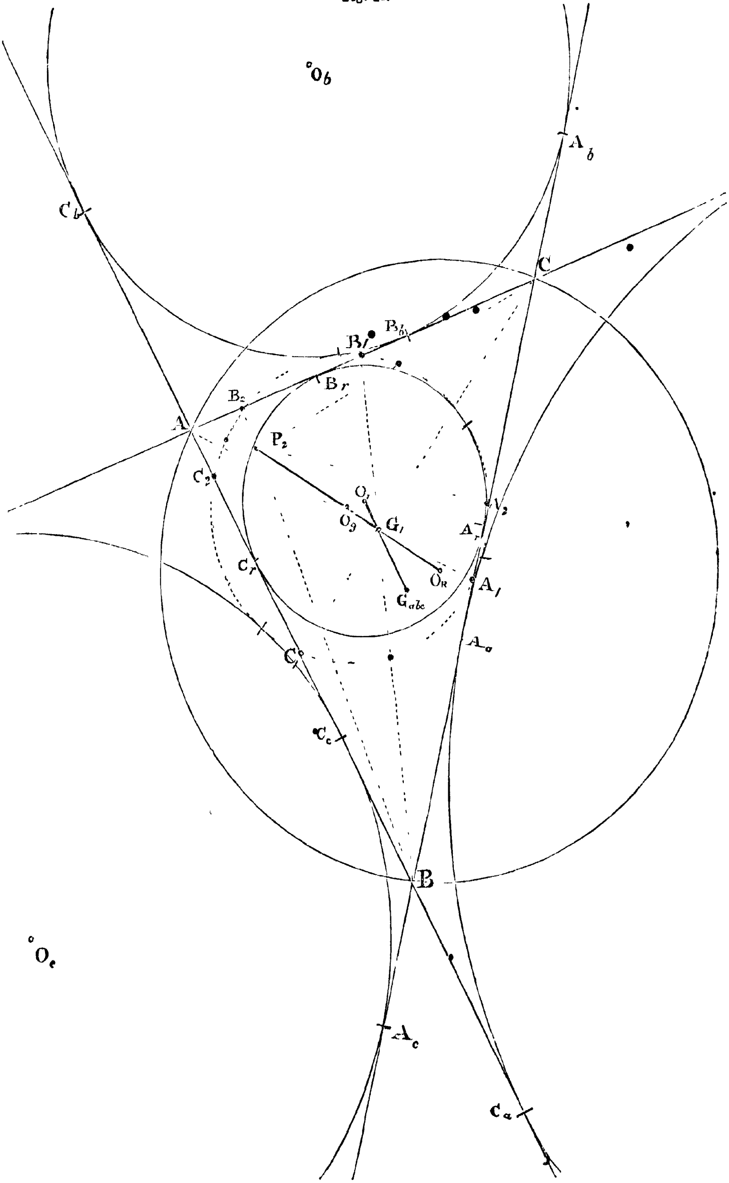
$$\frac{\bar{\alpha}}{\cos(B-C)} = \frac{\bar{\beta}}{\cos(C-A)} = \frac{\bar{\gamma}}{\cos(A-B)} = \frac{R}{2}. \quad (408)$$

259. *To find the equation of the circle which passes through* O_a, O_b, O_c (Art. 235).

Let the equation be assumed to be

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta - (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma) = 0. \quad (409)$$

Fig. 41.


O_b
A_b
C_b
C
B_l
B_b
B_r
B_2
A
P_2
C_2
O_1
O_g
G_1
A_2
A_r
A_1
O_R
G_abc
C_r
A_a
C_1
C_c
B
O_c
A_c
C_a

At O_a we have

$$-\alpha = \beta = \gamma = r_a.$$

Therefore $a-b-c-(a-b-c)(l-m-n) = 0$, by (409);

whence, $$l - m - n = 1.$$

Similarly, $$-l + m - n = 1,$$

and $$-l - m + n = 1.$$

These equations give

$$l = m = n = -1, \tag{410}$$

and, substituting these values for l, m, n in (409), we get

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (a\alpha + b\beta + c\gamma)(\alpha + \beta + \gamma) = 0 \tag{411}$$

for the equation of S_{abc}.

260. *To find the equation of the circle which passes through O_r, O_b, O_c.*

Proceeding as in the last example, and substituting in (409) the values of α, β, γ at the points O_r, O_b, O_c, successively, we get to determine l, m, n,

$$l + m + n = 1,$$

$$-l + m - n = 1,$$

and $$-l - m + n = 1:$$

whence,

$$-l = m = n = 1, \tag{412}$$

and we have for the equation of S_{rbc}

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (a\alpha + b\beta + c\gamma)(\alpha - \beta - \gamma) = 0. \tag{413}$$

261. The equations of some of the more important circles are collected in the subjoined list. For the sake of shortness $2S$ has been substituted for $a\alpha + b\beta + c\gamma$. For an explanation of the notation employed the reader is again referred to Art. 235.

(S_R). $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0.$

(S_r). $a\beta\gamma + b\gamma\alpha + c\alpha\beta - \frac{2S}{abc}\{a(s-a)^2\alpha + b(s-b)^2\beta + c(s-c)^2\gamma\} = 0.$

(S_a). $a\beta\gamma + b\gamma\alpha + c\alpha\beta - \frac{2S}{abc}\{a s^2 \alpha + b(s-c)^2\beta + c(s-b)^2\gamma\} = 0.$

(S_b). $a\beta\gamma + b\gamma\alpha + c\alpha\beta - \frac{2S}{abc}\{a(s-c)^2\alpha + b s^2\beta + c(s-a)^2\gamma\} = 0.$

(S_c). $a\beta\gamma + b\gamma\alpha + c\alpha\beta - \frac{2S}{abc}\{a(s-b)^2\alpha + b(s-a)^2\beta + c s^2\gamma\} = 0.$

(S_9). $a\beta\gamma + b\gamma\alpha + c\alpha\beta - S\{\cos A\,\alpha + \cos B\,\beta + \cos C\gamma\} = 0.$

(S_s). $a\beta\gamma + b\gamma\alpha + c\alpha\beta - 2S\{\cos A\,\alpha + \cos B\,\beta + \cos C\,\gamma\} = 0.$

(S_{abc}). $a\beta\gamma + b\gamma\alpha + c\alpha\beta + 2S(\alpha + \beta + \gamma) = 0.$

(S_{rbc}). $a\beta\gamma + b\gamma\alpha + c\alpha\beta + 2S(\alpha - \beta - \gamma) = 0.$

(S_{arc}). $a\beta\gamma + b\gamma\alpha + c\alpha\beta + 2S(\beta - \gamma - \alpha) = 0.$

(S_{abr}). $a\beta\gamma + b\gamma\alpha + c\alpha\beta + 2S(\gamma - \alpha - \beta) = 0.$

262. The radical axes of these circles taken in pairs, are represented (Art. 249) by the following equations:—

(S_R) and (S_r). $a(s-a)^2\alpha + b(s-b)^2\beta + c(s-c)^2\gamma = 0.$

(S_R) . . . (S_a). $as^2\alpha + b(s-b)^2\beta + c(s-c)^2\gamma = 0.$

(S_R) . . . (S_9). $\cos A\,\alpha + \cos B\,\beta + \cos C\,\gamma = 0.$

(S_R) . . . (S_s). $\cos A\,\alpha + \cos B\,\beta + \cos C\,\gamma = 0.$

(S_R) and (S_{abc}).	$\alpha + \beta + \gamma = 0.$
(S_R) . . . (S_{ibc}).	$\alpha - \beta - \gamma = 0.$
(S_r) . . . (S_a).	$\frac{b+c}{b-c} a\alpha + b\beta - c\gamma = 0.$
(S_r) . . . (S_9).	$\frac{a\alpha}{b-c} + \frac{b\beta}{c-a} + \frac{c\gamma}{a-b} = 0.$
(S_a) . . . (S_9).	$\frac{a\alpha}{b-c} + \frac{b\beta}{c+a} - \frac{c\gamma}{a+b} = 0.$
(S_b) . . . (S_9).	$\frac{-a\alpha}{b+c} + \frac{b\beta}{c-a} + \frac{c\gamma}{a+b} = 0.$
(S_c) . . . (S_9).	$\frac{a\alpha}{b+c} - \frac{b\beta}{c+a} + \frac{c\gamma}{a-b} = 0.$
(S_s) . . . (S_9).	$\cos A\alpha + \cos B\beta + \cos C\gamma = 0.$
(S_{abc}) . . . (S_{rbc}).	$\beta + \gamma = 0.$
(S_{abc}) . . . (S_{arc}).	$\gamma + \alpha = 0.$
(S_{abc}) . . . (S_{abr}).	$\alpha + \beta = 0.$
(S_{arc}) . . . (S_{abr}).	$\beta - \gamma = 0.$
(S_{abr}) . . . (S_{rbc}).	$\gamma - \alpha = 0.$
(S_{ibc}) . . . (S_{arc}).	$\alpha - \beta = 0.$

From the above it appears that S_R, S_s, *and* S_9 *are coaxal circles*, a conclusion to which we are also led by a comparison of the equation of S_9, which may be written in the form

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C - 2(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C) = 0, \quad (414)$$

with those of S_R and S_s (Eqq. 315, 320).

263. The co-ordinates of some of the principal points referred to in Art. 235 will be found below:—

(P_1) or (O_R). $$\frac{\bar{\alpha}}{\cos A} = \frac{\bar{\beta}}{\cos B} = \frac{\bar{\gamma}}{\cos C} = R.$$

(O_r). $$\bar{\alpha} = \bar{\beta} = \bar{\gamma} = r.$$

(O_a). $$-\bar{\alpha} = \bar{\beta} = \bar{\gamma} = r_a.$$

(O_b). $$\bar{\alpha} = -\bar{\beta} = \bar{\gamma} = r_b.$$

(O_c). $$\bar{\alpha} = \bar{\beta} = -\bar{\gamma} = r_c.$$

(O_9). $$\frac{\bar{\alpha}}{\cos(B-C)} = \frac{\bar{\beta}}{\cos(C-A)} = \frac{\bar{\gamma}}{\cos(A-B)} = \frac{R}{2}.$$

(P_2) or (O_s). $$\cos A\bar{\alpha} = \cos B\bar{\beta} = \cos C\bar{\gamma} = 2R\cos A\cos B\cos C.$$

(G_1). $$a\bar{\alpha} = b\bar{\beta} = c\bar{\gamma} = \frac{2S}{3}.$$

(G_r). $$a(s-a)\bar{\alpha} = b(s-b)\bar{\beta} = c(s-c)\bar{\gamma}.$$

(G_{abc}). $$\frac{a\bar{\alpha}}{s-a} = \frac{b\bar{\beta}}{s-b} = \frac{c\bar{\gamma}}{s-c} = 2r.$$

CHAPTER X.

GENERAL THEOREMS AND PROBLEMS.

264. *A conic is completely determined when five points upon it are given.*

For the most general form of the equation of a conic, viz.:

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2D\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta = 0, \qquad (415)$$

involves six constants, and the five equations which we obtain by successively substituting in (415) the co-ordinates of the given points are sufficient to determine their proportional values.

265. *A conic is completely determined when five tangents to it are given.*

The condition that any line (l, m, n) should touch the conic (415) is (Art. 131. (157).)

$$A'l^2 + B'm^2 + C'n^2 + 2D'mn + 2E'nl + 2F'lm = 0; \qquad (416)$$

and, if five tangents (l_1, m_1, n_1), (l_2, m_2, n_2), . . . be given, we shall have five such equations, whence the proportional values of A', B', C', . . . may be determined. These being known, the proportional values of the coefficients in (416)—the constituents of the determinant Δ, are also known. This proves the proposition.

266. *If from a point O two chords OR_1R_2, OS_1S_2 be drawn in given directions to a curve of the second degree, the ratio of the rectangles under the segments of the chords is the same for every position of O.*

For (Art. 13. (8).) the distance of O $(\alpha_0, \beta_0, \gamma_0)$ from the curve, in any direction, is given by the equation

$$\phi(\lambda, \mu, \nu)r^2 + \ldots. + \phi(\alpha_0, \beta_0, \gamma_0) = 0.$$

Hence, if $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$ are the two fixed directions, we have

$$\frac{OR_1 \,.\, OR_2}{OS_1 \,.\, OS_2} = \frac{\phi(\alpha_0, \beta_0, \gamma_0)}{\phi(\lambda_1, \mu_1, \nu_1)} : \frac{\phi(\alpha_0, \beta_0, \gamma_0)}{\phi(\lambda_2, \mu_2, \nu_2)}$$

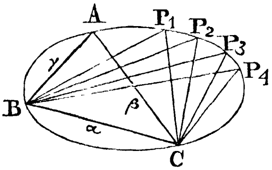
$$= \frac{\phi(\lambda_2, \mu_2, \nu_2)}{\phi(\lambda_1, \mu_1, \nu_1)},$$

an expression which is independent of α_0, β_0, γ_0, and is, therefore, constant for every position of O.

267. *To prove that the anharmonic ratio of the pencil formed by joining four points on a conic to any fifth point on it is constant.*

Let P_1, P_2, P_3, P_4 be the four given points, and A, B, C any other points on the conic.

Fig. 42.



Take the inscribed triangle ABC for the triangle of reference; then the equation of the conic (Art. 180. (N).) will be of the form

$$\frac{L}{\alpha} + \frac{M}{\beta} + \frac{N}{\gamma} = 0. \tag{417}$$

Let us assume

$$\begin{aligned} \gamma - h_1 \alpha &= 0, \\ \gamma - h_2 \alpha &= 0, \\ \gamma - h_3 \alpha &= 0, \\ \gamma - h_4 \alpha &= 0, \end{aligned} \qquad (418)$$

and

$$\begin{aligned} \beta - k_1 \alpha &= 0, \\ \beta - k_2 \alpha &= 0, \\ \beta - k_3 \alpha &= 0, \\ \beta - k_4 \alpha &= 0, \end{aligned} \qquad (419)$$

for the equations of the successive rays of the pencils

$\{B.\ P_1P_2P_3P_4\}$ and $\{C.\ P_1P_2P_3P_4\}$, respectively.

The point of intersection of the first pair of corresponding rays in (418) and (419) is given by the equations

$$\frac{\alpha}{1} = \frac{\beta}{k_1} = \frac{\gamma}{h_1}:$$

therefore, since P_1 is on the conic (417),

$$L + \frac{M}{k_1} + \frac{N}{h_1} = 0.$$

Hence $$h_1 = -\frac{Nk_1}{Lk_1 + M};$$

and we should get similar values for h_2, h_3, h_4, in terms of k_2, k_3, k_4, respectively.

Taking, therefore, one of the anharmonic ratios (Art. 92. (113).) of the first pencil, since

$$h_1 - h_2 = -\frac{MN}{(Lk_1 + M)(Lk_2 + M)}(k_1 - k_2)$$

etc. = etc.,

we get

$$\frac{h_1 - h_2}{h_3 - h_2} : \frac{h_1 - h_4}{h_3 - h_4} = \frac{k_1 - k_2}{k_3 - k_2} : \frac{k_1 - k_4}{k_3 - k_4}.$$

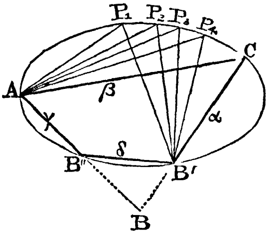
The equality of the other ratios may be shewn in a similar manner. We have therefore

$$\{B \, . \, P_1P_2P_3P_4\} = \{C \, . \, P_1P_2P_3P_4\},$$

and the truth of the proposition is established.

268. If the centres of the two pencils be made the opposite vertices of an inscribed quadrilateral, the proof of the Theorem of the last Article will be somewhat simplified.

Fig. 43.



Thus; let A, B' be the centres of the two pencils, ABC the triangle of reference, and $\delta = 0$ the equation of the fourth side $B'B''$ of the quadrilateral. The conic will be represented (Art. 208. (V).) by an equation of the form

$$\gamma\alpha = k\beta\delta. \qquad (420)$$

Let us assume for the equations of the corresponding rays of the two pencils

$$\left.\begin{aligned} \beta - l_1\gamma &= 0, \\ \beta - l_2\gamma &= 0, \\ \beta - l_3\gamma &= 0, \\ \beta - l_4\gamma &= 0; \end{aligned}\right\} \quad (421)$$

and

$$\left.\begin{aligned} \alpha - m_1\delta &= 0, \\ \alpha - m_2\delta &= 0, \\ \alpha - m_3\delta &= 0, \\ \alpha - m_4\delta &= 0; \end{aligned}\right\} \quad (422)$$

then, since they intersect on the conic, we get by substituting for α and β in (420),

$$\left.\begin{aligned} m_1 &= kl_1, \\ m_2 &= kl_2, \\ \text{etc.} &= \text{etc.}; \end{aligned}\right\} \quad (423)$$

and the second pencil is represented by the equations

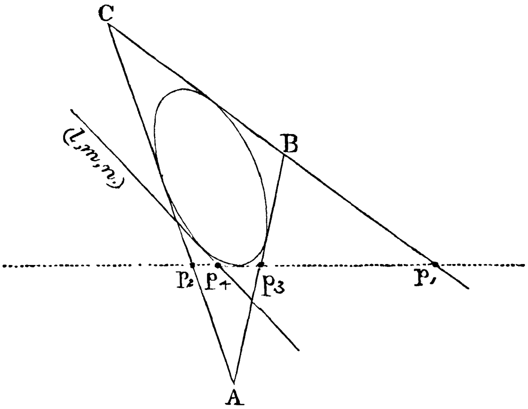
$$\begin{aligned} \alpha - kl_1\delta &= 0, \\ \alpha - kl_2\delta &= 0, \\ \alpha - kl_3\delta &= 0, \\ \alpha - kl_4\delta &= 0, \end{aligned}$$

and is therefore (Art. 93) homographic with (421).

269. *Four fixed tangents to a conic are cut by any variable tangent in points whose anharmonic ratios are constant.*

Let ABC, the triangle formed by any three of the fixed tangents, be taken as the triangle of reference.

Fig. 44.



The equation of the conic (Art. 187. (O_5).) will be of the form

$$(L\alpha)^{\frac{1}{2}} + (M\beta)^{\frac{1}{2}} + (N\gamma)^{\frac{1}{2}} = 0. \tag{424}$$

Suppose the equation of the fourth fixed tangent to be

$$l\alpha + m\beta + n\gamma = 0, \tag{425}$$

and let

$$\lambda\alpha + \mu\beta + \nu\gamma = 0 \tag{426}$$

represent any variable tangent meeting the four fixed ones in p_1, p_2, p_3, p_4.

The condition of tangency (Art. 212. (295).) gives

$$\frac{L}{l} + \frac{M}{m} + \frac{N}{n} = 0, \tag{427}$$

and $$\frac{L}{\lambda} + \frac{M}{\mu} + \frac{N}{\nu} = 0. \tag{428}$$

Now the successive rays of the pencil $\{A \cdot p_1 p_2 p_3 p_4\}$ may be readily shewn to have for their equations

$$\beta = 0,$$

$$(m\lambda - l\mu)\beta + (n\lambda - l\nu)\gamma = 0,$$

$$\gamma = 0,$$

and $$\mu\beta + \nu\gamma = 0,$$ respectively.

Hence (Art. 92. (113).), the ratio

$$\frac{p_1 p_2}{p_3 p_2} : \frac{p_1 p_4}{p_3 p_4} = \frac{k_1 - k_2}{k_3 - k_2} : \frac{k_1 - k_4}{k_3 - k_4}, \left[\text{when } k_1 = 0,\ k_2 = \frac{n\lambda - l\nu}{m\lambda - l\mu}, \right.$$

$$\left. k_3 = \infty,\ k_4 = \frac{\nu}{\mu}.\right]$$

$$= \left[\frac{k_2}{k_4}\right]_{\substack{k_2 = \frac{n\lambda - l\nu}{m\lambda - l\mu}. \\ k_4 = \frac{\nu}{\mu}.}}$$

$$= \frac{\mu(n\lambda - l\nu)}{\nu(m\lambda - l\mu)} = \frac{n\left(\frac{1}{l\nu} - \frac{1}{n\lambda}\right)}{m\left(\frac{1}{l\mu} - \frac{1}{m\lambda}\right)}$$

$$= -\frac{Mn}{Nm},$$ a constant quantity;

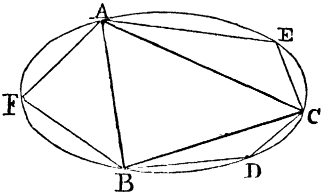
the last step following by reason of (427) and (428), which give by cross-multiplication

$$\frac{L}{\frac{1}{n\mu} - \frac{1}{m\nu}} = \frac{M}{\frac{1}{l\nu} - \frac{1}{n\lambda}} = \frac{N}{\frac{1}{m\lambda} - \frac{1}{l\mu}}.$$

270. *Pascal's Theorem.*—*The three pairs of opposite sides of a hexagon inscribed in a conic intersect in points which lie in the same straight line.*

Let $AECDBF$ be any inscribed hexagon, and let ABC be taken as the triangle of reference.

Fig. 45.



The equation of the conic will be (Art. 180. (N).) of the form

$$\frac{L}{\alpha} + \frac{M}{\beta} + \frac{N}{\gamma} = 0. \tag{429}$$

Suppose

$$\begin{array}{lll} (BD). & \left.\begin{array}{l}\alpha = k_1\gamma \\ \alpha = k_2\beta\end{array}\right\} & \begin{array}{l}\text{to be the equations of the two} \\ \text{sides which intersect in} \quad D,\end{array} \\ (CD). & & \\ (CE). & \left.\begin{array}{l}\beta = k_3\alpha \\ \beta = k_4\gamma\end{array}\right\} & \begin{array}{l}\ldots\ldots \\ \ldots\ldots \quad E,\end{array} \\ (AE). & & \\ (AF). & \left.\begin{array}{l}\gamma = k_5\beta \\ \gamma = k_6\alpha\end{array}\right\} & \begin{array}{l}\ldots\ldots \\ \ldots\ldots \quad F.\end{array} \\ (BF). & & \end{array}$$

Since D, E, F lie on the conic, we have from (429)

$$\begin{aligned} L + Mk_2 + Nk_1 &= 0, \\ Lk_3 + M + Nk_4 &= 0, \\ Lk_6 + Mk_5 + N &= 0, \end{aligned}$$

and, therefore,

$$\begin{vmatrix} 1, & k_2, & k_1 \\ k_3, & 1, & k_4 \\ k_6, & k_5, & 1 \end{vmatrix} = 0. \tag{430}$$

But the condition that the opposite sides should intersect on the same straight line,

$$l\alpha + m\beta + n\gamma = 0 \text{ (suppose)}, \tag{431}$$

is

$$\begin{vmatrix} 1, & k_3, & k_6 \\ k_2, & 1, & k_5 \\ 1, & k_3, & k_6 \end{vmatrix} = 0. \tag{432}$$

For the intersection on (431) of

$$CE \text{ and } BF \text{ gives } l + mk_3 + nk_6 = 0;$$

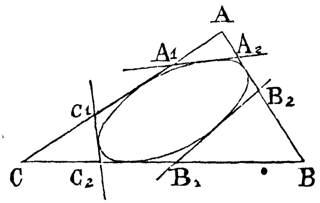
$$\text{that of } AF \ldots CD \ldots lk_2 + m + nk_5 = 0;$$

$$\ldots\ BD \ldots AE \ldots lk_1 + mk_4 + n = 0.$$

But the conditions (430) and (432) are identical; hence the theorem is proved[a].

271. *Brianchon's Theorem.—The diagonals which connect the three pairs of opposite angles of a hexagon described about a conic meet in a point.*

Fig. 46.



Let the circumscribed triangle *ABC* be taken as the triangle of reference; then the equation of the conic (Art. 187. (O_5).) will be of the form

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[a] This proof of Pascal's Theorem is given by Mr. Ferrers in his treatise on Trilinear Co-ordinates.

$$(L\alpha)^{\frac{1}{2}} + (M\beta)^{\frac{1}{2}} + (N\gamma)^{\frac{1}{2}} = 0. \tag{433}$$

Let A_1A_2, B_1B_2, C_1C_2, the other three sides of the hexagon, have for their equations

$$\begin{aligned} &l_1\alpha + m_1\beta + n_1\gamma = 0, \\ &l_2\alpha + m_2\beta + n_2\gamma = 0, \\ \text{and} \quad &l_3\alpha + m_3\beta + n_3\gamma = 0, \end{aligned} \tag{434}$$

respectively.

It may be proved by the usual method (e.g see Art. 98) that the equations of the three diagonals B_2C_1, A_2C_2, A_1B_1 are, respectively,

$$\begin{aligned} \alpha + \frac{m_2}{l_2}\beta + \frac{n_3}{l_3}\gamma &= 0, \\ \frac{l_1}{m_1}\alpha + \beta + \frac{n_3}{m_3}\gamma &= 0, \\ \frac{l_1}{n_1}\alpha + \frac{m_2}{n_2}\beta + \gamma &= 0. \end{aligned} \tag{435}$$

Also, since (434) are tangents to the conic (433), we have

$$\frac{L}{l_1} + \frac{M}{m_1} + \frac{N}{n_1} = 0,$$

$$\frac{L}{l_2} + \frac{M}{m_2} + \frac{N}{n_2} = 0,$$

$$\frac{L}{l_3} + \frac{M}{m_3} + \frac{N}{n_3} = 0;$$

and, therefore,

$$\begin{vmatrix} \frac{1}{l_1}, & \frac{1}{m_1}, & \frac{1}{n_1} \\ \frac{1}{l_2}, & \frac{1}{m_2}, & \frac{1}{n_2} \\ \frac{1}{l_3}, & \frac{1}{m_3}, & \frac{1}{n_3} \end{vmatrix} = 0; \qquad (436)$$

but this may be written in the form

$$\begin{vmatrix} \frac{1}{l_1}, & \frac{1}{l_2}, & \frac{1}{l_3} \\ \frac{1}{m_1}, & \frac{1}{m_2}, & \frac{1}{m_3} \\ \frac{1}{n_1}, & \frac{1}{n_2}, & \frac{1}{n_3} \end{vmatrix} = 0,$$

or,

$$\begin{vmatrix} 1, & \frac{m_2}{l_2}, & \frac{n_3}{l_3} \\ \frac{l_1}{m_1}, & 1, & \frac{n_3}{m_3} \\ \frac{l_1}{n_1}, & \frac{m_2}{n_2}, & 1 \end{vmatrix} = 0,$$

which (Art. 55. (60).) is the condition that (435) should meet in a point. This proves the theorem.

272. *To find the locus of the centre of a conic which touches three given straight lines and passes through the given point* $(\alpha_1, \beta_1, \gamma_1)$.

Let the given tangents be taken as the triangle of reference, and suppose the equation of the conic to be

$$(L\alpha)^{\frac{1}{2}} + (M\beta)^{\frac{1}{2}} + (N\gamma)^{\frac{1}{2}} = 0.$$

The centre (Art. 213. (308).) is given by the equations

$$\frac{\bar{\alpha}}{Nb + Mc} = \frac{\bar{\beta}}{Lc + Na} = \frac{\bar{\gamma}}{Ma + Lb} = -\frac{1}{k}\ \text{(say)};$$

whence,

$$\begin{aligned} & Mc + Nb + \bar{\alpha}k = 0, \\ Lc \quad & + Na + \bar{\beta}k = 0, \end{aligned}$$

and

$$Lb + Ma \quad + \bar{\gamma}k = 0,$$

which give

$$\frac{L}{\begin{vmatrix} c, & b, & \bar{\alpha} \\ 0, & a, & \bar{\beta} \\ a, & 0, & \bar{\gamma} \end{vmatrix}} = \frac{M}{-\begin{vmatrix} 0, & b, & \bar{\alpha} \\ c, & a, & \bar{\beta} \\ b, & 0, & \bar{\gamma} \end{vmatrix}} = \frac{N}{\begin{vmatrix} 0, & c, & \bar{\alpha} \\ c, & 0, & \bar{\beta} \\ b, & a, & \gamma \end{vmatrix}}$$

or,

$$\frac{L}{a(-a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma})} = \frac{M}{b(a\bar{\alpha} - b\bar{\beta} + c\bar{\gamma})} = \frac{N}{c(a\bar{\alpha} + b\bar{\beta} - c\bar{\gamma})}. \tag{437}$$

But, since $(\alpha_1, \beta_1, \gamma_1)$ always lies on the curve,

$$(L\alpha_1)^{\frac{1}{2}} + (M\beta_1)^{\frac{1}{2}} + (N\gamma_1)^{\frac{1}{2}} = 0; \tag{438}$$

and, substituting in (438) for L, M, N from (437) and writing α, β, γ for $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$, we get for the equation of the locus

$$\begin{aligned} \sqrt{\{a\alpha_1(-a\alpha + b\beta + c\gamma)\}} + \sqrt{\{b\beta_1(a\alpha - b\beta + c\gamma)\}} \\ + \sqrt{\{c\gamma_1(a\alpha + b\beta - c\gamma)\}} = 0. \end{aligned} \tag{439}$$

Now,

$$-a\alpha + b\beta + c\gamma = 0,$$

$$a\alpha - b\beta + c\gamma = 0,$$

and
$$a\alpha + b\beta - c\gamma = 0,$$

represent the sides of the triangle $A_1 B_1 C_1$ whose vertices are the middle points of the sides of the triangle of reference. The required locus, therefore, is a conic section which touches the three sides of the triangle $A_1B_1C_1$.

273. *To find the equation of the polar reciprocal of the conic*

$$\phi_1(\alpha, \beta, \gamma) = A_1\alpha^2 + B_1\beta^2 + C_1\gamma^2 + 2D_1\beta\gamma + 2E_1\gamma\alpha + 2F_1\alpha\beta = 0,$$

with respect to the conic

$$\phi_2(\alpha, \beta, \gamma) = A_2\alpha^2 + B_2\beta^2 + C_2\gamma^2 + 2D_2\beta\gamma + 2E_2\gamma\alpha + 2F_2\alpha\beta = 0.$$

Let $(\alpha', \beta', \gamma')$ be a point on the reciprocal curve. Its polar with regard to $\phi_2(\alpha, \beta, \gamma) = 0$ has for its equation (Art. 134)

$$\left(\frac{d\phi_2}{d\alpha'}\right)\alpha + \left(\frac{d\phi_2}{d\beta'}\right)\beta + \left(\frac{d\phi_2}{d\gamma'}\right)\gamma = 0,$$

and the condition that it should be a tangent to $\phi_1(\alpha, \beta, \gamma) = 0$ is, therefore (Art. 131. (155).),

$$\begin{vmatrix} A_1, & F_1, & E_1, & \left(\frac{d\phi_2}{d\alpha'}\right) \\ F_1, & B_1, & D_1, & \left(\frac{d\phi_2}{d\beta'}\right) \\ E_1, & D_1, & C_1, & \left(\frac{d\phi_2}{d\gamma'}\right) \\ \left(\frac{d\phi_2}{d\alpha'}\right), & \left(\frac{d\phi_2}{d\beta'}\right), & \left(\frac{d\phi_2}{d\gamma'}\right), & 0 \end{vmatrix} = 0.$$

Hence, suppressing the accents, we have for the equation of the polar reciprocal

$$\begin{vmatrix} A_1, & F_1, & E_1, & \left(\frac{d\phi_2}{d\alpha}\right) \\ F_1, & B_1, & D_1, & \left(\frac{d\phi_2}{d\beta}\right) \\ E_1, & D_1, & C_1, & \left(\frac{d\phi_2}{d\gamma}\right) \\ \left(\frac{d\phi_2}{d\alpha}\right), & \left(\frac{d\phi_2}{d\beta}\right), & \left(\frac{d\phi_2}{d\gamma}\right), & 0 \end{vmatrix} = 0; \qquad (440)$$

or, to use the notation hitherto employed,

$$\phi_1\left[\left(\frac{d\phi_2}{d\alpha}\right), \left(\frac{d\phi_2}{d\beta}\right), \left(\frac{d\phi_2}{d\gamma}\right)\right]' = 0. \qquad (441)$$

EXAMPLES.

Throughout the following examples, except when the contrary is stated, ABC is used to denote the triangle of reference.

I.

Find the proportional values (Art. 26) of the co-ordinates of the following points; ABC being the triangle of reference.

1. The middle point of the angle-bisector AA_0.

If l be the length of AA_0, the actual values of the co-ordinates will be $\frac{l(b+c)}{2a}\sin\frac{A}{2}$, $\frac{l}{2}\sin\frac{A}{2}$, $\frac{l}{2}\sin\frac{A}{2}$, and their proportional values $(b+c,\ a,\ a)$.

2. The middle point of the side-bisector AA_1.

The actual co-ordinates are $\frac{S}{a}$, $\frac{a}{4}\sin C$, $\frac{a}{4}\sin B$, and the point may be represented by $\left(\frac{2}{a}, \frac{1}{b}, \frac{1}{c}\right)$.

3. The middle point of the perpendicular AA_2.

Ans. $(1, \cos C, \cos B)$.

4. The point of intersection of the side-bisector BB_1 with the angle-bisector CC_0.

The equations of these lines are, respectively, $c\gamma - a\alpha = 0$ and $\alpha - \beta = 0$. Hence the actual values of the co-ordinates of their intersection are $a\alpha = a\beta = c\gamma = \frac{2aS}{2a+b}$, and their proportional values $\left(\frac{1}{a}, \frac{1}{a}, \frac{1}{c}\right)$.

5. The intersection of the external bisectors of the angles B and C.

Ans. $(-1, 1, 1)$.

II.

6. Shew that the three perpendiculars drawn to the sides of an equilateral triangle, from any point within it, are together equal to the altitude of the triangle.

7. Prove that

$$A_1 \sin A + B_1 \sin B + C_1 \sin C = 0;$$

where A_1, B_1, C_1 are the areas of the triangles formed by joining the vertices of the triangle ABC to the centres of the inscribed and escribed circles.

8. Shew that the sum of the reciprocals of the distances of G_1 (Art. 263) from the sides of the triangle of reference is equal to three times the radius of the inscribed circle.

For the notation employed in the following examples the reader is referred to Art. 235 and fig. 41.

9. Prove that O_9 is the middle point of P_2O_R (Art. 257).

Let α_9, α_2, α_R be, respectively, the α- co-ordinates of the three given points. Then, by Art. 263,

$$\begin{aligned} \alpha_R + \alpha_2 &= R\cos A + 2R\cos B\cos C \\ &= R\cos(B - C) \\ &= 2\alpha_9; \end{aligned}$$

and the same is true, by symmetry, for the α- and β- co-ordinates.

10. Prove that $G_1P_2 = 2G_1O_R$ (Art. 257).

11. Shew that $G_1G_{abc} = 2G_1O_r$.

12. The sides of the triangle of reference being 5, 12, 13; construct (Art. 15) the line whose equation is

$$2\alpha - 3\beta + 4\gamma = 0.$$

III.

Write down, in their *symmetrical* forms, the equations of the following right lines (Arts. 7, 8, 9):—

13. The straight line through A which biscots the $\angle BAC$ of the triangle of reference.

The θ, ϕ and ψ of this line are, respectively, $\frac{A}{2} + C$, $\frac{A}{2}$, and $\frac{A}{2}$; its direction-cosines, therefore, (since, by Art. 8[a], they are equal to $-\sin\theta$, $\sin\phi$, $\sin\psi$) are

$$-\sin\left(\frac{A}{2} + C\right), \sin\frac{A}{2}, \sin\frac{A}{2}.$$

Also it passes through $A\left(\frac{2S}{a}, 0, 0\right)$. Hence the required equations are

$$\frac{\alpha - \frac{2S}{a}}{-\sin\left(\frac{A}{2} + C\right)} = \frac{\beta}{\sin\frac{A}{2}} = \frac{\gamma}{\sin\frac{A}{2}}.$$

14. The straight line through A perpendicular to BC.

Its θ, ϕ and ψ are, respectively, $\frac{\pi}{2}, \frac{\pi}{2} - C, \frac{\pi}{2} - B$; its direction-cosines, therefore, are $-\sin\frac{\pi}{2}$, $\sin\left(\frac{\pi}{2} - C\right)$, $\sin\left(\frac{\pi}{2} - B\right)$; and, since it passes through the point $\left(\frac{2S}{a}, 0, 0\right)$, its equations are

$$\frac{\alpha - \frac{2S}{a}}{-1} = \frac{\beta}{\cos C} = \frac{\gamma}{\cos C}.$$

15. The straight line perpendicular to AB through its middle point.

Its direction-cosines are $\cos B$, $\cos A$, $-\sin\frac{\pi}{2}$, and its equations, therefore,

$$\frac{\alpha - \frac{c}{2}\sin B}{\cos B} = \frac{\beta - \frac{c}{2}\sin A}{\cos A} = \frac{\gamma}{-1}.$$

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[a] When the direction-cosines of a straight line through one of the vertices are sought, the signs of the denominators (Art. 8) are best determined by considering the segments of some straight line parallel to the given one.

16. The straight line through C parallel to AB.

The direction-cosines are $-\sin B$, $\sin A$, $\sin 0$, and its equations, therefore, are

$$\frac{\alpha}{-\sin B} = \frac{\beta}{\sin A} = \frac{\gamma - \dfrac{2S}{c}}{0}.$$

17. The straight line through B at right angles to AB.

Ans. $$\frac{\alpha}{\cos B} = \frac{\beta - \dfrac{2S}{b}}{\cos A} = \frac{\gamma}{1}.$$

IV.

Find, in the homogeneous form, the equations of the following right lines:—

18. The straight line through C parallel to AB.

The equation must be capable of expression in either of the forms $\alpha + k_1\beta = 0$ (Art. 16) and $(a\alpha + b\beta + c\gamma) - k_2\gamma = 0$ (Art. 67): identifying these, we get $k_2 = c$, and for the equation required, $a\alpha + b\beta = 0$.

19. The sides of the triangle DEF, formed by joining the points in which the angle-bisectors meet the opposite sides of the Δ of reference.

First method.—The bisectors (Art. 19) are $\beta - \gamma = 0$, $\gamma - \alpha = 0$, $\alpha - \beta = 0$. The equation of EF, since it passes through the intersection of $\gamma - \alpha = 0$ and $\beta = 0$, must be of the form $\gamma - \alpha + k_1\beta = 0$; also of the form $\alpha - \beta + k_2\gamma = 0$, since it passes through the intersection of $\alpha - \beta = 0$ and $\gamma = 0$. Identifying these, we have $k_1 = 1$, $k_2 = -1$, and, for the equation of EF

$$-\alpha + \beta + \gamma = 0.$$

Similarly, $$\alpha - \beta + \gamma = 0,$$

and $$\alpha + \beta - \gamma = 0,$$

are the equations of FD, DE, respectively.

Second method.—The co-ordinates of D, E, F are, respectively, proportional to $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$. Hence the equations of the joining lines may be written down at once from eq. 58. Art. 68.

20. The straight line which joins the intersection of

$$2a\alpha + b\beta + c\gamma = 0 \text{ and } b\beta - c\gamma = 0$$

with that of

$$2b\alpha + a\beta + c\gamma = 0 \text{ and } a\beta - c\gamma = 0.$$

Ans. $\dfrac{ab}{a+b}(\alpha - \beta) + c\gamma = 0.$

21. The straight line through A at right angles to AB.

First method.—Take any point P on this line, and draw PQ perpendicular to CA. Then, from the geometry,

$$\frac{\beta}{\gamma} = \frac{-PQ}{PA} = \frac{-PA\sin PAQ}{PA} = -\cos A.$$

Therefore $$\beta + \cos A\gamma = 0;$$

which is the required equation.

Second method.—Its direction-cosines are

$$\cos B,\ \cos A,\ -1;$$

and it passes through the point A, whose co-ordinates are proportional to

$$1,\ 0,\ 0;$$

hence, Art. 59. (71) gives for the required equation

$$\begin{vmatrix} \cos A, & -1 \\ 0, & 0 \end{vmatrix}\alpha + \begin{vmatrix} -1, & \cos B \\ 0, & 1 \end{vmatrix}\beta + \begin{vmatrix} \cos B, & \cos A \\ 1, & 0 \end{vmatrix}\gamma = 0,$$

or $$\beta + \cos A\gamma = 0.$$

Third method.—The equation must be of the form $\beta + k\gamma = 0$, since the line passes through A. And, since it is $\perp^r$ to $\gamma = 0$, we have by Art. 72, $k - \cos A = 0$.

$$\therefore\ \beta + \cos A\gamma = 0.$$

22. The straight line which joins B_2, C_2, the feet of the perpendiculars from B and C upon the opposite sides.

The co-ordinates of B_2, C_2 are proportional to

$$\cos C,\quad 0,\quad \cos A,$$

and

$$\cos B,\ \cos A,\quad 0;$$

hence Art. 58 gives for the required equation

$$\begin{vmatrix} 0, & \cos A \\ \cos A, & 0 \end{vmatrix} \alpha + \begin{vmatrix} \cos A, & \cos C \\ 0, & \cos B \end{vmatrix} \beta + \begin{vmatrix} \cos C, & 0 \\ \cos B, & \cos A \end{vmatrix} \gamma = 0,$$

or,

$$\cos A\alpha - \cos B\beta - \cos C\gamma = 0.$$

23. The straight line which joins the middle points of the sides CA, AB. Also shew that it is parallel to BC.

Let the equation be $l\alpha + m\beta + n\gamma = 0$: since (Art. 1) it passes through the points $\left(\frac{b}{2}\sin C,\ 0,\ \frac{b}{2}\sin A\right)$ and $\left(\frac{c}{2}\sin B,\ \frac{c}{2}\sin A,\ 0\right)$, we have

$$l \sin C + n \sin A = 0,$$

and

$$l \sin B + m \sin A = 0;$$

and the required equation is

$$-\sin A\alpha + \sin B\beta + \sin C\gamma = 0,$$

or

$$-a\alpha + b\beta + c\gamma = 0.$$

Also, since it may be written in the form $2a\alpha = a\alpha + b\beta + c\gamma$, or $\alpha = \frac{S}{a}$, it appears (Art. 67) that the straight line which it represents is parallel to $BC\ (\alpha = 0)$.

24. The three perpendiculars to the sides at their middle points. Also shew that they meet in a point.

First method.—Let A_1 be the middle point of BC and D the point in which the perpendicular at D meets the second side (CA suppose) of the Δ. Join DB. Then evidently $DB = DC = l$ (say), and the co-ordinates of D are $l \sin C$, 0, $l \sin (B - C)$. Hence the points A_1 and D may be represented by

$$[0,\quad \sin C,\quad \sin B]$$

and

$$[\sin C,\quad 0,\quad \sin (B - C)] \quad \text{respectively.}$$

The equation of the $\perp'$ is therefore, by Art. 58,

$$\begin{vmatrix} \sin C, & \sin B \\ 0, & \sin(B-C) \end{vmatrix} \alpha + \begin{vmatrix} \sin B, & 0 \\ \sin(B-C), & \sin C \end{vmatrix} \beta + \begin{vmatrix} 0, & \sin C \\ \sin C, & 0 \end{vmatrix} \gamma = 0,$$

or, dividing by sin C,

$$\sin(B-C)\alpha + \sin B\beta - \sin C\gamma = 0.$$

Similarly for the other two perpendiculars we have

$$\sin(C-A)\beta + \sin C\gamma - \sin A\alpha = 0,$$

$$\sin(A-B)\gamma + \sin A\alpha - \sin B\beta = 0.$$

These equations, when multiplied by $\sin^2 A$, $\sin^2 B$, $\sin^2 C$, respectively, and added together, vanish. The lines, therefore, meet in a point (Art. 17).

Second Method.—The first $\perp^r$ passes through the point $(0, \sin C, \sin B)$, and its direction-cosines are $(-1, \cos C, \cos B)$: hence, by Art. 59, we have for its equation

$$\begin{vmatrix} \sin C, & \sin B \\ \cos C, & \cos B \end{vmatrix} \alpha + \begin{vmatrix} \sin B, & 0 \\ \cos B, & -1 \end{vmatrix} \beta + \begin{vmatrix} 0, & \sin C \\ -1, & \cos C \end{vmatrix} \gamma = 0,$$

or,

$$\sin(B-C)\alpha + \sin B\beta - \sin C\gamma = 0.$$

V.

For the notation and method employed in this set of examples, the student is referred to Arts. 26, 58, 263, *and to* Fig. 41.

25. Find the equation of the straight line which joins O_R and O_r.

It passes through $(\cos A, \cos B, \cos C)$ and $(1, 1, 1)$, and its equation is

$$(\cos B - \cos C)\alpha + (\cos C - \cos A)\beta + (\cos A - \cos B)\gamma = 0.$$

26. Shew that the equation of the straight line which joins P_2 and O_R is

$$\sin 2A \sin(B-C)\alpha + \sin 2B \sin(C-A)\beta + \sin 2C \sin(A-B)\gamma = 0.$$

27. Find the equation of $O_9 G_1$.

Ans. $$\left[\frac{\cos(A-B)}{\sin B} - \frac{\cos(C-A)}{\sin C}\right]\alpha + \&c. \ldots\ldots = 0,$$

which is easily reduced to the form of the result of the last example; hence it appears that O_R, G_1, O_9 and P_2 are collinear (Art. 257).

28. Find the equation of $O_r O_9$.

Ans. $$[\cos(A-B) - \cos(C-A)]\,\alpha + \&c. \ldots\ldots = 0.$$

29. Shew that the equation of $G_1 O_r$ is

$$\left(\frac{1}{b} - \frac{1}{c}\right)\alpha + \left(\frac{1}{c} - \frac{1}{a}\right)\beta + \left(\frac{1}{a} - \frac{1}{b}\right)\gamma = 0.$$

30. Find the equation of $G_1 G_{abc}$, and shew that O_r, G_1 and G_{abc} are collinear (Art. 257).

Ans. The result is the same as that of the last example.

31. Find the equation of $G_r\, G_{abc}$.

Ans. $$a^2(b-c)(s-a)\,\alpha + b^2(c-a)(s-b)\,\beta + c^2(a-b)(s-c)\gamma = 0,$$

or, $$a^2(\cos B - \cos C)\,\alpha + b^2(\cos C - \cos A)\,\beta + c^2(\cos A - \cos B)\gamma = 0.$$

32. Find the equation of the straight line which joins the centres of the circumscribed and self-conjugate circles.

Ans. $$\sin 2A \sin(B-C)\alpha + \sin 2B \sin(C-A)\beta + \sin 2C \sin(A-B)\gamma = 0.$$

33. Shew that the straight line which joins the centre of the inscribed circle with that of an escribed circle passes through a vertex of the triangle.

Ans. The equation of $O_r\, O_a$ is $\beta - \gamma = 0$, which is also the equation of the bisector of the angle at A.

VI.

34. In any triangle *ABC*, the internal bisector of one angle and the external bisector of the two other angles, meet in a point.

Their equations are $\beta - \gamma = 0$, $\gamma + \alpha = 0$, $\alpha + \beta = 0$, and if the last be subtracted from the sum of the other two the result is $= 0$. Hence (Art. 17), the proposition is true.

35. Find the equation of the line through A which is perpendicular to $\beta - k\gamma = 0$.

It must be of the form $\beta + \lambda\gamma = 0$ by Art. 16, and the condition of Art. 72 gives $1 - k\lambda - (k - \lambda)\cos A = 0$; whence $\lambda = \dfrac{1 - k\cos A}{k - \cos A}$, and the required equation is

$$(k - \cos A)\beta + (1 - k\cos A)\gamma = 0.$$

36 Find the equation of the straight line through the centre of the inscribed circle and perpendicular to *AB*.

Its direction-cosines are $\cos B, \ \cos A, \ -1$, and the co-ordinates of the centre are proportional to $1, \ 1, \ 1$.

The required equation is, therefore, by Art. 59,

$$(1 + \cos A)\alpha - (1 + \cos B)\beta + (\cos B - \cos A)\gamma = 0.$$

37. Straight lines are drawn through the vertices of *ABC*, the triangle of reference, and cutting the straight lines

$$\frac{\beta}{b} + \frac{\gamma}{c} = 0, \ \frac{\gamma}{c} + \frac{\alpha}{a} = 0, \ \frac{\alpha}{a} + \frac{\beta}{b} = 0,$$

at right angles. Find their equations and shew that they meet in a point.

First Method.—The direction-cosines of a perpendicular to $\dfrac{\beta}{b} + \dfrac{\gamma}{c} = 0$, are, by Eq. 84 of Art. 70,

$$-c\cos C - b\cos B, \ c - b\cos A, \ b - c\cos A;$$

or,

$$-c\cos C - b\cos B, \quad a\cos B, \quad a\cos C.$$

Also it passes through A (1, 1, 1). Hence Art. 59 gives for its equation

$$\frac{\beta}{\cos B} - \frac{\gamma}{\cos C} = 0.$$

And the equations of the other two perpendiculars, viz.

$$\frac{\gamma}{\cos C} - \frac{\alpha}{\cos A} = 0$$

and $$\frac{\alpha}{\cos A} - \frac{\beta}{\cos B} = 0,$$

may be written down by symmetry. These three equations vanish identically when added together. The lines, therefore, meet in a point.

Second Method.—The equation of a straight line through A must be of the form $\beta - k\gamma = 0$, and if it be perpendicular to $\frac{\beta}{b} + \frac{\gamma}{c} = 0$, we shall have by the condition of Art. 72 $\frac{1}{b} - \frac{k}{c} - \left(\frac{1}{c} - \frac{k}{b}\right)\cos A = 0$; whence

$$k = \frac{c - b\cos A}{b - c\cos A} = \frac{a\cos B}{a\cos C},$$ and the equation is

$$\frac{\beta}{\cos B} - \frac{\gamma}{\cos C} = 0, \text{ as before.}$$

38. Find the equation of the straight line passing through the two points defined by the equations

$$l_1\alpha = m_1\beta = n_1\gamma,$$

$$l_2\alpha = m_2\beta = n_2\gamma.$$

Ans. $l_1 l_2(m_1 n_2 - m_2 n_1)\alpha + m_1 m_2(n_1 l_2 - n_2 l_1)\beta + n_1 n_2(l_1 m_2 - l_2 m_1)\gamma = 0.$

39. Within a triangle two points are taken. The distance of the first from any side is proportional to that side. The distance of the second from any side is proportional to the sum of the other two sides. Shew that the straight line joining these two points passes through the centre of the circle inscribed in the triangle.

40. If the straight line $l\alpha + m\beta + n\gamma = 0$ be perpendicular to $\alpha = 0$, prove that $l = m\cos C + n\cos B$.

41. Prove that perpendiculars drawn to the sides of a triangle from the centres of the escribed circles meet in a point.

42. If through the vertices of any triangle there be drawn three straight lines meeting in a point, the three lines drawn through the same vertices and equally inclined to the bisectors of the angles will also meet in a point.

43. Two straight lines are drawn from the vertex A; one through the middle point of BB_1 (one of the bisectors of sides), the other parallel to BB_1. Find their equations and shew that they are Harmonic conjugates with respect to the sides AB, AC.

The equation of BB_1 (Art. 22) is $c\gamma - a\alpha = 0$. Hence the equation of the parallel through $A\,(1, 0, 0)$ is $b\beta + 2c\gamma = 0$. The middle point of BB_1 is $\left(\frac{b}{4}\sin C, \frac{S}{b}, \frac{b}{4}\sin A\right)$, or $\left(\frac{1}{a}, \frac{2}{b}, \frac{1}{c}\right)$. Therefore the equation of the first line (Art. 58) is $b\beta - 2c\gamma = 0$: and the condition of Art. 95 is satisfied.

44. Two triangles ABC, XYZ, have their sides parallel; viz. YZ to BC, etc. . . . : shew that AX, BY, CZ meet in a point.

Let the equations of the sides of XYZ be $\alpha - k_1 = 0$, $\beta - k_2 = 0$, $\gamma - k_3 = 0$ respectively (Art. 67). The equation of AX must be of the form $\beta - \lambda\gamma = 0$, and also of the form $\beta - k_2 - \mu(\gamma - k_3) = 0$. Identifying these we get $\mu = \lambda = \frac{k_2}{k_3}$; and the equation becomes

$$\frac{\beta}{k_2} - \frac{\gamma}{k_3} = 0.$$

Similarly, we get
$$\frac{\gamma}{k_3} - \frac{\alpha}{k_1} = 0$$

and
$$\frac{\alpha}{k_1} - \frac{\beta}{k_2} = 0$$

for the equations of BY, CZ, respectively, and these three equations when added together vanish identically.

45. Two triangles ABC, XYZ, are *homologous*, i.e. are such that their corresponding sides intersect on the same straight line (called the *axis of*

homology); shew that the straight lines which join corresponding vertices meet in a point.

Let $$l\alpha + m\beta + n\gamma = 0$$

represent the axis of homology. We may assume

$$\lambda\alpha + m\beta + n\gamma = 0,$$

$$l\alpha + \mu\beta + n\gamma = 0,$$

$$l\alpha + m\beta + \nu\gamma = 0,$$

for the equations of YZ, ZX and XY, respectively; and the equations of AX, BY, CZ will be, respectively,

$$(m - \mu)\beta - (n - \nu)\gamma = 0,$$

$$(n - \nu)\gamma - (l - \lambda)\alpha = 0,$$

$$(l - \lambda)\alpha - (m - \mu)\beta = 0;$$

and these when added together vanish identically.

The student will observe that the last example is a particular case of the present, the line at infinity being in that case the axis of homology. The centre of homology in Ex. 44 is given by the equations

$$\frac{\alpha}{k_1} = \frac{\beta}{k_2} = \frac{\gamma}{k_3} = \frac{2S}{ak_1 + bk^2 + ck_3};$$

in this example by the equations

$$(l - \lambda)\alpha = (m - \mu)\beta = (n - \nu)\gamma.$$

46. From the angles A, B, C of a triangle, straight lines are drawn, through a point O, to meet the opposite sides in E, F, G, respectively. FG, GE, EF are produced to meet BC, CA, AB, respectively, in P, Q, R. Prove that P, Q, R lie in one line.

Taking ABC for the Δ of reference and assuming $(\alpha_0, \beta_0, \gamma_0)$ for the co-ordinates of 0, we shall have

$$\frac{\beta}{\beta_0} = \frac{\gamma}{\gamma_0}, \quad \frac{\gamma}{\gamma_0} = \frac{\alpha}{\alpha_0}, \quad \frac{\alpha}{\alpha_0} = \frac{\beta}{\beta_0},$$

for the equations of the three lines through the vertices, and it will be easily shewn that P, Q, R lie on the right line whose equation is

$$\frac{\alpha}{\alpha_0} + \frac{\beta}{\beta_0} + \frac{\gamma}{\gamma_0} = 0.$$

47. If three straight lines, drawn through the vertices of a triangle, meet in a point, their respective parallels, drawn through the middle points of the opposite sides, also meet in a point.

48. A triangle LMN is formed by drawing, through each vertex of the triangle of reference, a straight line which makes an angle θ with the bisector of the angle at that vertex. Find the equations of the sides.

The direction-cosines of the bisector of the $\angle A$ are

$$\cos\left(\frac{\pi}{2} + \frac{A}{2} + B\right), \cos\left(\frac{\pi}{2} - \frac{A}{2}\right), \cos\left(\frac{3\pi}{2} + \frac{A}{2}\right);$$

hence, by Art. 32, those of MN are

$$\cos\left(\frac{\pi}{2} + \theta + \frac{A}{2} + B\right), \cos\left(\frac{\pi}{2} + \theta - \frac{A}{2}\right), \cos\left(\frac{3\pi}{2} + \theta + \frac{A}{2}\right),$$

$$\text{or,} \quad -\sin\left(\theta + \frac{A}{2} + B\right), -\sin\left(\theta - \frac{A}{2}\right), \sin\left(\theta + \frac{A}{2}\right).$$

Therefore the equation of MN is, by Art. 59, since the co-ordinates of A are proportional to 1, 0, 0,

$$\sin\left(\theta + \frac{A}{2}\right)\beta + \sin\left(\theta - \frac{A}{2}\right)\gamma = 0.$$

Similarly,
$$\sin\left(\theta + \frac{B}{2}\right)\gamma + \sin\left(\theta - \frac{B}{2}\right)\alpha = 0,$$

and
$$\sin\left(\theta + \frac{C}{2}\right)\alpha + \sin\left(\theta - \frac{C}{2}\right)\beta = 0,$$

are the equations of the sides NL, LM.

49. Equilateral triangles are described on the sides of the triangle ABC, and their vertices joined to the opposite sides of the triangle. Find the equations of the joining lines and shew that they meet in a point.

The co-ordinates of the vertex of the Δ described on BC are proportional to $\sin\frac{\pi}{3}$, $\sin\left(\frac{\pi}{3}+C\right)$, $\sin\left(\frac{\pi}{3}+B\right)$, therefore, by Art. 58, the equation of the straight line which joins this vertex to A (1, 0, 0) is

$$\sin\left(\frac{\pi}{3}+B\right)\beta-\sin\left(\frac{\pi}{3}+C\right)\gamma=0.$$

Similarly,
$$\sin\left(\frac{\pi}{3}+C\right)\gamma-\sin\left(\frac{\pi}{3}+A\right)\alpha=0,$$

$$\sin\left(\frac{\pi}{3}+A\right)\alpha-\sin\left(\frac{\pi}{3}+B\right)\beta=0,$$

are the equations of the other two lines. The three, therefore, meet in a point

$$\sin\left(\frac{\pi}{3}+A\right)\alpha=\sin\left(\frac{\pi}{3}+B\right)\beta=\sin\left(\frac{\pi}{3}+C\right)\gamma.$$

VII.

50. Through the vertex A of the triangle ABC pairs of lines are drawn equally inclined to the sides AB, AC; shew that they form a pencil in involution of which the internal and external bisectors of the angle A are the double lines.

Such a pair of lines may be represented by equations of the form

$$\beta-k_1\gamma=0,\quad k_1\beta-\gamma=0;$$

these may also be written in the form

$$(\beta+\gamma)+\frac{1+k}{1-k}(\beta-\gamma)=0,$$

$$(\beta+\gamma)-\frac{1+k}{1-k}(\beta-\gamma)=0.$$

Hence (Art. 95), they are harmonic conjugates with respect to the two bisectors, and the rest follows from Art. 121.

51. The equation of a conic being $\beta\gamma-k\alpha^2=0$; prove that if the point $(\beta\gamma)$ be joined to the points where any tangent meets the curve and the

line $\alpha = 0$, the joining lines form with the lines $\beta = 0$, $\gamma = 0$ a harmonic pencil.

See Art. 210. (*R*).

52. Chords drawn to a conic through a fixed point O without the curve meet the curve in the points P_1, Q_1; P_2, Q_2; P_3, Q_3; etc. . . Shew that the pencil formed by joining these points to the point of contact of one of the tangents that may be drawn through O forms a system in involution of which that tangent and the polar of O are the double lines.

(See Art. 225.)

53. Find the locus of the pole of a fixed right line with respect to the system of conics represented by $LM = kR^2$.

54. Shew that any pair of conjugate diameters of a hyperbola are harmonic conjugates with respect to the asymptotes of the curve.

Let the equation of the hyperbola (Art. 205) be $\beta\gamma = k^2$, and let

$$\frac{\alpha - \alpha_0}{\lambda} = \frac{\beta - \beta_0}{\mu} = \frac{\gamma - \gamma_0}{\nu} = r$$

be the equations of a chord. The diameter parallel to this is evidently $\frac{\beta}{\mu} = \frac{\gamma}{\nu}$, or $\frac{\beta}{\mu} - \frac{\gamma}{\nu} = 0$. Putting $\beta = \beta_0 + \mu r$, $\gamma = \gamma_0 + \nu r$ in the equation of the curve, we get

$$\mu\nu r^2 + (\beta_0\nu + \gamma_0\mu)r + \beta_0\gamma_0 - k^2 = 0.$$

Hence, for the locus of middle points we get

$$\beta\nu + \gamma\mu = 0;$$

and

$$\frac{\beta}{\mu} + \frac{\gamma}{\nu} = 0$$

is the equation of the conjugate diameter, This (Art. 95) proves the proposition.

55. The bisectors of the angles of the triangle *ABC* meet the conic

$$\frac{L}{\alpha} + \frac{M}{\beta} + \frac{N}{\gamma} = 0$$

in the points *D, E, F*; find the equations of *BD, CD, DE.*

Ans. $$\gamma + \frac{M+N}{L}\alpha = 0,$$

$$\frac{M+N}{L}\alpha + \beta = 0,$$

$$(M+N)\alpha + (N+L)\beta - n\gamma = 0.$$

56. A conic is described touching the sides of the triangle ABC in D, E, F; if P, Q, R are the points in which the straight lines AD, BE, CF, respectively, meet the conic, shew that BR and CQ intersect on AD, CP and AR on BE, and AQ and BP on CF.

Vide Art. 187 et sqq.

57. Find the condition that the equation

$$L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0$$

may represent a rectangular hyperbola, and hence prove that every rectangular hyperbola described about a given triangle passes through the point of intersection of the perpendiculars from the vertices of the triangle upon the opposite sides.

Ans. The condition required is

$$L\cos A + M\cos B + N\cos C = 0;$$

and the co-ordinates of the point P_2 (Art. 263) evidently satisfy the equation of the conic.

58. Find the equation of the conic which touches the sides of the triangle of reference ABC at their middle points.

Ans. $(a\alpha)^{\frac{1}{2}} + (b\beta)^{\frac{1}{2}} + (c\gamma)^{\frac{1}{2}} = 0.$

59. Shew that the equation of the fourth common tangent to the inscribed circle of the triangle ABC and that escribed on BC is

$$\cos\frac{A}{2}\alpha + \sin\frac{B-C}{2}(\beta - \gamma) = 0.$$

60. A conic is inscribed in a triangle ABC, the vertices of which are joined to a given point O by straight lines AO, BO, CO cutting the opposite sides in a, b, c, respectively: find the equations of the tangents (other than the sides of the triangle) which can be drawn to the conic from the points a, b, c.

Ans. If $(L\alpha)^{\frac{1}{2}} + (M\beta)^{\frac{1}{2}} + (N\gamma)^{\frac{1}{2}} = 0$ be the equation of the conic, the equation of the second tangent from a will be

$$Lmn\alpha + (Nm - Mn)m\beta + (Mn - Nm)n\gamma = 0.$$

61. Find the equation of the ellipse which touches the sides of the triangle of reference where they are met by the bisectors of the opposite angles.

Comparing eqq. 238 (Art. 187) with the equations of Art. 19 we find, $L = M = N$; the required equation, therefore, is

$$\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} = 0.$$

62. If three conics have a common chord, the other three common chords meet in a point.

Let $S - tu = 0$, $S - tv = 0$, $S - tw = 0$ (Art. 169) represent the three conics; $t = 0$ being their common chord. The equations of their other three chords of intersection are $v - w = 0$, $w - u = 0$, $u - v = 0$; and the straight lines represented by these equations evidently meet in a point.

63 The asymptotes of a hyperbola are tangents to an ellipse; shew that the chords which join the points of intersection of the two curves are parallel.

Take the asymptotes of the hyperbola and the chord of contact of the ellipse for the sides of the triangle of reference. Then (Art. 208) the equations of the two curves will be of the forms

$$\beta\gamma = l^2\alpha^2,\ \beta\gamma = m^2.$$

The equation $l^2\alpha^2 - m^2 = 0$, therefore represents chords of intersection, and from its form it appears (Art. 67) that these chords are parallel to one another and to the chord of contact $\alpha = 0$.

64. If the angle A of the triangle of reference be a right angle, the equation

$$\beta^2 + \gamma^2 = k^2\alpha^2$$

represents a conic which has $\alpha = 0$ for its directrix and A for its focus.

For $\beta^2 + \gamma^2$ represents the square of the distance of any point on the locus from A, and the equation expresses that the distance of any point on the locus from A is in a constant ratio to its distance from BC.

65. The tangents at the extremities of any focal chord meet in the corresponding directrix.

Assume the equation of the conic in the form given in the last example, taking the given focal chord for one side ($\gamma = 0$, say) of the triangle of reference. It appears as in Art. 195 that

$$\gamma + k\alpha = 0, \; \gamma - k\alpha = 0$$

are tangents at the extremities of this chord, and the form of their equations shews that they intersect on $\alpha = 0$.

66. Find the locus of the pole of the line

$$l\alpha + m\beta + n\gamma = 0$$

with respect to a conic which passes through the four fixed points $\alpha_1, \pm\beta_1, \pm\gamma_1$.

Ans. $$\frac{l\alpha_1^2}{\alpha} + \frac{m\beta_1^2}{\beta} + \frac{c\gamma_1^2}{\gamma} = 0.$$

67. Find the locus of the centre of a conic which passes through the same four points.

Ans. $$\frac{a\alpha_1^2}{\alpha} + \frac{b\beta_1^2}{\beta} + \frac{c\gamma_1^2}{\gamma} = 0.$$

68. Find the locus of the pole of the straight line

$$l\alpha + m\beta + n\gamma = 0$$

with respect to a conic passing through three points and touching the fixed line

$$A\alpha + B\beta + C\gamma = 0.$$

Ans. $$\sqrt{\{A(m\beta + n\gamma - l\alpha)\alpha\}} + \sqrt{\{B(n\gamma + l\alpha - m\beta)\beta\}} + \sqrt{\{C(l\alpha + m\beta - n\gamma)\gamma\}} = 0.$$

69. Obtain the equation of the chord which joins the two points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ on the conic $\beta\gamma - k\alpha = 0$ in the form

$$\left[a\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) + \frac{kb}{\gamma_1\gamma_2}\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right) + \frac{kc}{\beta_1\beta_2}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right)\right]k\alpha$$

$$= \left[a\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right) + \frac{2kc}{\beta_1\beta_2}\right]\beta + \left[a\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) + \frac{2kb}{\gamma_1\gamma_2}\right]\gamma = 0.$$

70. If a quadrilateral be inscribed in a circle, the product of the perpendicular distances of any point on the circumference from two opposite sides will be equal to the product of its perpendicular distances from the other two sides.

Let $$\gamma\alpha = k\beta(l\alpha + m\beta + n\gamma)$$

[Art. 208. (v).] be the equation of a conic circumscribed about a quadrilateral whose sides are $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and $l\alpha + m\beta + n\gamma = 0$.

If it be a circle, we shall have (Art. 119)

$$c(mc - nb) = \frac{ac}{k} = a(ma - lb)$$

or, $$\frac{mc - nb}{a} = \frac{1}{k} = \frac{bl - ma}{-c} = \frac{b(an - cl)}{c^2 - a^2}.$$

$$\therefore \frac{1}{k^2} = \frac{a(an - cl)(bl - am) + b(bl - am)(cm - bn) + c(cm - bn)(an - cl)}{-\frac{ca}{b}(c^2 - a^2) - abc + \frac{ca}{b}(c^2 - a^2)}$$

$$= l^2 + m^2 + n^2 - 2mn\cos A - 2ln\cos B - 2lm\cos C;$$

and $\frac{1}{k} = \{l, m, n\}$.

Hence the equation of the circumscribed circle is $\gamma\alpha = \dfrac{(l\alpha + m\beta + n\gamma)\beta,}{\{l, m, n\}}$ and (Art. 47) the proposition is proved.

71. If three conics have each double contact with a fourth, their six chords of intersection will pass, three by three, through the same points.

We may assume the equations of the three conics to be $S - u^2 = 0$, $S - v^2 = 0$, $S - w^2 = 0$ (Art. 169). Their chords of intersection will be represented by the equations

$$v^2 - w^2 = 0, \quad w^2 - u^2 = 0, \quad u^2 - v^2 = 0.$$

Thus we have four groups of three equations each representing chords which meet in a point: viz.—

$$v + w = 0, \quad w - u = 0, \quad u + v = 0,$$

$$v - w = 0, \quad w + u = 0, \quad u + v = 0,$$

$$v + w = 0,\ w + u = 0,\ u - v = 0,$$

$$v - w = 0,\ w - u = 0,\ u - v = 0.$$

The student will observe that the above theorem includes that of Brianchon, of which an independent proof was given in Art. 271.

72. In any triangle the bisector of any angle and the straight line which is perpendicular to the opposite side at its middle point, intersect on the circumference of the circumscribing circle.

For the point of intersection of $\alpha - \beta = 0$ and $\sin A\alpha - \sin B\beta + \sin(A - B)\gamma = 0$, is given by the equations

$$\alpha = \beta = -\gamma\frac{\sin(A - B)}{\sin A - \sin B}.$$

whence we may easily shew that

$$\frac{\sin A}{\alpha} + \frac{\sin B}{\beta} + \frac{\sin C}{\gamma} = 0.$$

73. Shew that if the tangents to a circumscribed ellipse at the vertices of the triangle of reference are parallel to the opposite sides the equation of the conic is

$$\frac{\beta\gamma}{a} + \frac{\gamma\alpha}{b} + \frac{\alpha\beta}{c} = 0.$$

74. The two pairs of tangents at the extremities of the diagonals of a quadrilateral inscribed in a conic, and the pairs of opposite sides intersect on the same straight line.

This Theorem is easily deduced from that of Pascal which was proved in Art. 270.

75. Shew that if a conic cut two sides of a triangle in points equidistant from their middle points it will cut the third side in the same way.

Let the sides BC, CA of the triangle of reference be cut in this manner; then if $\phi(\alpha,\beta,\gamma) = 0$ be the equation of the conic, $(\alpha_0, \beta_0, \gamma_0)$ the middle point of BC, and λ, μ, ν the direction-cosines of that side, we shall have by equation (8) of Art. 13, since the roots are equal and of opposite sign

$$\left(\frac{d\phi}{d\alpha_0}\right)\lambda + \left(\frac{d\phi}{d\beta_0}\right)\mu + \left(\frac{d\phi}{d\gamma_0}\right)\nu = 0,$$

which, since the value of α_0, β_0, γ_0 are 0, $\frac{a}{2}\sin C$, $\frac{a}{2}\sin B$, and those of λ, μ, ν are 0, $-\sin C$, $\sin B$, gives

$$(B\sin C + D\sin B)\sin C - (D\sin C + C\sin B)\sin B = 0;$$

that is,

$$\frac{B}{\sin^2 B} - \frac{C}{\sin^2 C} = 0.$$

Similarly, since CA is cut in the same way, we have

$$\frac{C}{\sin^2 C} - \frac{A}{\sin^2 A} = 0.$$

These give also

$$\frac{A}{\sin^2 A} - \frac{B}{\sin^2 B} = 0;$$

but this is the condition that the middle point of the side AB should be the bisection of the intercept on that side.

76. If $\frac{l_1}{\alpha} + \frac{m_1}{\beta} + \frac{n_1}{\gamma} = 0$ and $\frac{l_2}{\alpha} + \frac{m_2}{\beta} + \frac{n_2}{\gamma} = 0$ be the equations of two conics described about the triangle of reference, find the equations of the several lines joining the centre of the inscribed circle with the four points of intersection of the two conics.

77. With the angular points of a triangle ABC as centres, and the sides as asymptotes, three hyperbolas are described, having P, Q, R for their respective vertices; prove that if

$$AP\sin\frac{A}{2} = BQ\sin\frac{B}{2} = CR\sin\frac{C}{2},$$

the intersection of each pair of hyperbolas lies on the axis of the third.

The given condition may evidently be stated otherwise; thus,—the perpendicular distance of the vertex of each hyperbola from its asymptotes is the same.

If we assume for the equations of the three curves $\beta\gamma = l^2$, $\gamma\alpha = m^2$, $\alpha\beta = n^2$, so that their axes are $\beta - \gamma = 0$, $\gamma - \alpha = 0$, $\alpha - \beta = 0$; the above condition gives $l = m = n$.

The three equations are, therefore,

$$\beta\gamma = l^2,$$

$$\gamma\alpha = l^2,$$

$$\alpha\beta = l^2;$$

and, since the first may be written in the form

$$\gamma\alpha - l^2 - (\alpha - \beta)\,\gamma = 0$$

it is clear that it passes through the intersection of the second with the axis of the third.

78. Find the polar reciprocal of the conic

$$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 = 0$$

with respect to the conic

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 = 0.$$

Ans. $$\frac{l^4\alpha^2}{L^2} + \frac{m^4\beta^2}{M^2} + \frac{n^4\gamma^2}{N^2} = 0.$$

79. Find the polar reciprocal of the conic

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2D\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta = 0$$

with respect to the conic

$$\alpha^2 + \beta^2 + \gamma^2 = 0.$$

Ans. $$\begin{vmatrix} 0, & \alpha, & \beta, & \gamma \\ \alpha, & A, & F, & E \\ \beta, & F, & B, & D \\ \gamma, & E, & D, & C \end{vmatrix} = 0.$$

80. An equilateral hyperbola is described with regard to which a given triangle is self-conjugate. Shew that the curve passes through the centres of the inscribed and escribed circles of the triangle.

The co-ordinates of the four centres (Art. 263) will satisfy the equation

$$L\alpha^2 + M\beta^2 + N\gamma^2 = 0$$

provided that

$$L + M + N = 0;$$

but this (Art. 150) is the condition that the curve should be a rectangular hyperbola.

81. Three circles mutually touching each other are described about the vertices of the triangle of reference. Shew that the three straight lines which join the centre of one circle to the point of contact of the other two meet in a point.

Let x, y, z be the radii of the circles described about A, B, C, respectively. The point of contact of the circles about B and C is $(0, z\sin C, y\sin B)$, and the equations of the three lines are

$$y\sin B\beta - z\sin C\gamma = 0,$$

$$z\sin C\gamma - x\sin A\alpha = 0,$$

$$x\sin A\alpha - y\sin B\beta = 0;$$

which vanish identically when added together.

82. Supposing that α, β, γ are the perpendiculars from any point in the plane of the triangle ABC upon the straight lines which join its vertices to the centre of the inscribed circle, prove that

$$\alpha\cos\frac{A}{2} + \beta\cos\frac{B}{2} + \gamma\cos\frac{C}{2} = 0.$$

83. Shew that the radius of the nine-point circle of any triangle is one-half that of the circumscribed circle.

84. The circle which passes through the extremities of any side of a triangle and through the centre of the inscribed circle, passes also through the centre of the circle escribed on that side; and its centre lies on the circumference of the circumscribed circle.

85. In any triangle the centres O_b, O_c, and the vertices B, C lie on the circumference of the same circle; and the centre of this circle lies on the circumference of the circumscribed circle.

86. If x, y, z be the distances of the centre of the nine-point circle from the vertices of the triangle of reference, its distance from the intersection of perpendiculars from the vertices, and R the radius of the circumscribed circle, shew that

$$x^2 + y^2 + z^2 = \frac{5}{2} R^2 - p^2.$$

THE END.

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